ON PROXIMALLY OPEN MAPPINGS OF METRIC SPACES

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Abstract

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ON PROXIMALLY OPEN MAPPINGS OF METRIC SPACES

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Proximally open mappings $f : P \to Q$ were introduced in ([6]) as mappings preserving δ-neighborhoods of sets*, i.e. satisfying the condition: $A \subset B \subset P \Rightarrow fA \subset fB$. In the same work the study of such mappings was begun. Here we prove that the equi-open image** of a metrizable proximity space is metrizable; if, moreover, the mapping is compact, the image of a metrizable complete ([8]) space is complete.

The notions given below were apparently first used in ([2]).

Definitions. Sequences x_n, y_n of points of a proximity space are called **equivalent** if, for every increasing sequence of indices n_k, one has $\{x_{n_k}\}\delta\{y_{n_k}\}$. A cover $\mathfrak{M}$ is called **quasiuniform***** if, whatever equivalent sequences x_n and y_n may be, for infinitely many indices i one has $x_i \in \mathrm{St}(y_i, \mathfrak{M})$.

Lemma 1. *Let $f : M \to P$ be an equi-open mapping of a proximity space, $\mathfrak{U}$ a uniform cover of M, $\mathfrak{U}^f = \{\mathrm{St}(f^{-1}p, \mathfrak{U}) \mid p \in P\}$, and $\mathfrak{M} = f\mathfrak{U}^f$. Then $\mathfrak{M}$ is a quasiuniform cover of the space P.*

Proof. Suppose the contrary. Then there exist equivalent sequences $x_i \sim y_i$ in the space P and $\forall i : \ x_i \notin \mathrm{St}(y_i, \mathfrak{M})$. Denote $R_i = f^{-1}x_i$, $S_i = f^{-1}y_i$, and let $\mathfrak{B}$ be a uniform cover star-refining $\mathfrak{U}$. We assert that each $\mathrm{St}(R_i, \mathfrak{B})$ intersects only finitely many sets S_j. Indeed, if the set $J_i = \{j \mid S_j \cap \mathrm{St}(R_i, \mathfrak{B}) \neq \emptyset\}$ were infinite (for some i), then, in view of

$$A_i = \bigcup\{S_j \cap \mathrm{St}(R_i, \mathfrak{B}) \mid j \in J_i\} \subset \mathrm{St}(R_i, \mathfrak{U}) = L \in \mathfrak{U}^f,$$

one would also have $fA_i = \{y_j \mid j \in J_i\} \subset fL \in \mathfrak{M}$; but $\{x_j \mid j \in J_i\}\delta\{y_j \mid j \in J_i\}$, and therefore $\{x_j \mid j \in J_i\} \cap fL \neq \emptyset$, which contradicts our assumptions about the cover $\mathfrak{M}$. Similarly, all sets

$$J'_i = \{j \mid R_j \cap \mathrm{St}(S_i, \mathfrak{B}) \neq \emptyset\}$$

are finite.

Now one can—in the same way as this was done in ([2])—select distant cofinal parts from the sequences R_i and S_i. To this end define the sequence $\Theta = \{i_n\}$, putting $i_0 = 0$,

$$i_{n+1} = \min\Big(N \setminus \bigcup_{k \le i_n} (J_k \cup J_k')\Big).$$

Put $R = \bigcup\{R_i \mid i \in \Theta\}$ and $S = \bigcup\{S_i \mid i \in \Theta\}$; then $R\bar{\delta}S$, since, obviously,

$$R \cap \mathrm{St}(S, \mathfrak{B}) = \emptyset.$$

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* Proximally open, or equi-open, mappings are not analogous to uniformly open mappings in the sense of ([4]), i.e. satisfying the requirement $\mathfrak{U} \in \mathfrak{U}_X$: $f\,\mathrm{St}(x, \mathfrak{U}) \supset \mathrm{St}(fx, \mathfrak{B})$. The point is that in the case of proximity spaces it is not at all clear whether, for instance in Levin–Raiskov' s sense, the restriction $F \mid F^{-1}Q$ of an open mapping $F : X \to \bar{Q}$ of a compact X onto a compactification Q (even if the proximity spaces Q and $P = F^{-1}Q$ are metrizable) is open. On the other hand, extensions $\bar{f} : \bar{P} \to \bar{Q}$ of equi-open mappings $f : P \to Q$ are open (see ([6])).

** All mappings under consideration are proximally continuous.

*** For the definition of a **uniform** cover of a proximity space see, for example, ([7]). It can be shown that every uniform cover is quasiuniform.

Thus, $R \subset M \setminus S$. Therefore $fR \subset f(M \setminus S) = P \setminus fS$, i.e. $fR\delta fS$, or $\{x_i \mid i \in \Theta\}\delta\{y_i \mid i \in \Theta\}$. This contradicts the fact that the sequences x_i and y_i are equivalent.

Theorem 1. *If the proximity space M is metrizable and the mapping $f : M \to P$ is equi-open, then the space P is also metrizable.*

Proof. Let

$$\mathfrak{D}_n^{-1} = \left\{O\left(z, \frac{1}{n}\right) \mid z \in M\right\}$$

be the "exact $1/n$-covering" of M, and let $\mathfrak{R}_n = f\mathfrak{D}^f_{n^{-1}}$. By the preceding lemma, all the coverings $\mathfrak{R}_n$ are quasi-uniform; obviously,

$$\mathfrak{R}_1 > \mathfrak{R}_2 > \cdots.$$

In [5] S. Leader proved that a proximity space is metrizable if and only if it has a decreasing system of quasi-uniform coverings separating any two sets precisely when they are far apart. We shall now verify that this condition is satisfied for the sequence $\mathfrak{R}_n$.

Let $A\delta B$; then, in view of the openness of f, $f^{-1}A\delta f^{-1}B$, and for every n

$$f^{-1}A \cap \mathrm{St}(f^{-1}A, \mathfrak{D}^{f}_{n^{-1}}) \neq \emptyset,$$

whence it follows that

$$A \cap \mathrm{St}(B, \mathfrak{R}_n) \neq \emptyset.$$

Conversely, suppose $A\bar{\delta}B$. There exists $O \subset P$ with $A \subset O$ and $O\bar{\delta}B$; therefore also

$$f^{-1}A \subset f^{-1}O \quad \text{and} \quad f^{-1}O\bar{\delta}f^{-1}B.$$

For some m one has simultaneously

$$\mathrm{St}(f^{-1}A, \mathfrak{D}_{m^{-1}}) \subset f^{-1}O \quad \text{and} \quad \mathrm{St}(f^{-1}B, \mathfrak{D}_{m^{-1}}) \cap f^{-1}O = \emptyset.$$

Therefore, whatever point $p \in P$ is chosen, the set

$$\mathrm{St}(f^{-1}p, \mathfrak{D}_{m^{-1}})$$

does not meet one of the sets $f^{-1}A$ or $f^{-1}B$. Indeed, if

$$\mathrm{St}(f^{-1}p, \mathfrak{D}_{m^{-1}}) \cap f^{-1}A \neq \emptyset,$$

then

$$\mathrm{St}(f^{-1}A, \mathfrak{D}_{m^{-1}}) \cap f^{-1}p \neq \emptyset;$$

thus $f^{-1}p \cap f^{-1}O \neq \emptyset$, hence $f^{-1}p \subset fO$. Thus,

$$f^{-1}A \cap \mathrm{St}(f^{-1}B, \mathfrak{D}^{f}_{m^{-1}}) = \emptyset,$$

and consequently

$$A \cap \mathrm{St}(B, f\mathfrak{D}^{f}_{m^{-1}}) = \emptyset,$$

i.e.

$$A \cap \mathrm{St}(B, \mathfrak{R}_m) = \emptyset.$$

Corollary. *If $f : M \to P$ is an equi-open mapping of metric spaces and $\mathfrak{U}$ is a uniform covering of M, then $(f\mathfrak{U})^{**}$ is a uniform covering of P.*

Proof. From the proof of the cited theorem of Leader it follows that, in metrizable proximity spaces, the star of a quasi-uniform covering is a uniform covering; note also that

$$f\mathfrak{U}^f = (f\mathfrak{U})^*.$$

Lemma 2. *Let the mapping of proximity spaces $f : M \to P$ be equi-open and compact, let $a_i \in M$ be a countable sequence of points, $x_i = fa_i \in P$ a Cauchy sequence, $\mathfrak{B}$ a uniform covering in M, and $V_i = f\,\mathrm{St}(a_i, \mathfrak{B})$. There exist indices i and j for which*

$$U_i \supset \{x_r \mid r > j\}.$$

Proof. Suppose the contrary. Let $\mathfrak{W}$ be a uniform covering triply star-inscribed in $\mathfrak{B}$, and let

$$W_i = f\,\mathrm{St}(a_i, \mathfrak{W}).$$

First we show (incidentally, only here shall we use the compactness of the mapping!) that every infinite intersection of sets W_i is empty. Indeed, suppose, to the contrary, that

$$p \in \bigcap\{W_{i_t} \mid t = 1, 2, ...\}.$$

Then for every t one can find points

$$q_t \in f^{-1}p \cap \mathrm{St}(a_{i_t}, \mathfrak{W}).$$

The set $\{q_t\}$ has a point of contact q, and therefore the set

$$Q = \{q_t\} \cap \mathrm{St}(q, \mathfrak{W})$$

is infinite. It is true that

$$\mathrm{St}(q, \mathfrak{W}^*) \supset \mathrm{St}(Q, \mathfrak{W});$$

thus the set $\mathrm{St}(q, \mathfrak{W}^*)$ contains infinitely many elements a_{i_t}—denote their totality by A. Now, if $a \in A$, then

$$q \in \mathrm{St}(a, \mathfrak{W}^*)$$

and therefore

$$\mathrm{St}(a, \mathfrak{B}) \supset \mathrm{St}(\mathrm{St}(a, \mathfrak{W}^*), \mathfrak{W}^*) \supset \mathrm{St}(q, \mathfrak{W}^*) \supset A.$$

Hence

$$V = f\,\mathrm{St}(a, \mathfrak{B}) \supset fA.$$

Choose H with

$$V \supset H \supset fA.$$

The set $\{x_i\} \setminus H$ is far from the subsequence fA, and therefore is finite; hence

$$V \supset H \supset \{x_r \mid r > \mathrm{const}\}.$$

Now, assuming that the lemma is false, for every i find an index $j(i) > i$ with $x_{j(i)} \notin V_i$; put $y_i = x_{j(i)}$, and let

$$R_i = f^{-1} y_i.$$

It is asserted that, whatever i may be, $\mathrm{St}(a_i, \mathfrak{W})$ meets only finitely many of the sets R_k: indeed, if

$$R_k \cap \mathrm{St}(a_{i^0}, \mathfrak{W}) \neq \emptyset$$

for all $k \in \{k\}$, then, in view of

$$\mathrm{St}(a^{i^0}, \mathfrak{W}) \subset \mathrm{St}(a_{i^0}, \mathfrak{B}),$$

it follows that

$$\{y_k\} \subset W_{i^0} \subset V_{i^0},$$

and therefore,

if $W_{i_0} \subset K \subset V_{i_0}$, in view of $y_k \sim x_k \sim x$, it follows that $x_i \in K$ for almost all i—contrary to our assumption.

We shall show that, analogously to the preceding argument, the set $\mathrm{St}(R_i, \mathfrak{B})$, whatever i may be, contains only finitely many points a_k. If not, then for infinitely many indices k (with $i = \mathrm{const}$) one has $\mathrm{St}(a_k, \mathfrak{B}) \cap R_i \neq \emptyset$, and hence $\bigcap W_k \neq \emptyset$, which, as we have already established, is impossible.

We now construct the sequences a_{i_s} and R_{i_s} in the usual way: $i_0 = 0$, and i_{s+1} is the least index, starting from which $a_i \notin \mathrm{St}(R_{i_s}, \mathfrak{B})$ and

$$R_i \cap \mathrm{St}(a_{i_s}, \mathfrak{B}) = \emptyset.$$

It follows from the construction, evidently, that

$$\{a_{i_s} \mid s\} \cap \mathrm{St}(\bigcup\{R_{i_s} \mid s\}, \mathfrak{B}) = \emptyset$$

and, therefore,

$$\{a_{i_s} \mid s\} \subset M \setminus \bigcup\{R_{i_s} \mid s\};$$

further, in view of the proximal openness of f,

$$\{fa_{i_s} \mid s\} = \{x_{i_s} \mid s\} \subset f(M \setminus \bigcup\{R_{i_s} \mid s\}) = P \setminus f(\bigcup\{R_{i_s} \mid s\}) = P \setminus \{y_{i_s} \mid s\},$$

i.e. $\{x_{i_s}\} \, \delta \, \{y_{i_s}\}$. But both these sequences are subsequences of the fundamental sequence x_i, and therefore cannot be far apart!

Theorem 2. Let M be a metrizable proximity space, and let the mapping $f : M \to P$ be compact and equi-open. The space P is complete if M is complete.

Proof. Let $x_i \in P$ be an arbitrary fundamental sequence. Put

$$S_i = \bigcup\{f^{-1}x_j \mid j \geq i\}$$

and consider the system

$$\mathfrak{Z} = \{\mathrm{St}(S_i, \mathcal{U}) \mid i, \mathcal{U}\}$$

(the covers $\mathcal{U}$ are uniform). We assert that $\mathfrak{Z}$ is a stable system in Isbell' s sense ([3]), i.e.

$$\forall \mathcal{U} : \bigcap\{\mathrm{St}(Z, \mathcal{U}) \mid Z \in \mathfrak{Z}\} \in \mathfrak{Z}.$$

We first show that, for any fixed uniform cover $\mathfrak{B}$ of the space M, there exists an n with

$$S_n \subset \mathrm{St}(S_m, \mathfrak{B})$$

for whatever m. Suppose this is not so, and that, in particular, for arbitrarily large n there are infinitely many m with

$$\bigcap_m (f^{-1}x_n \setminus \mathrm{St}(f^{-1}x_m, \mathfrak{B})) \neq \emptyset.$$

Number the chosen n' s by indices r and fix points

$$a_r \in \bigcap_m (f^{-1}x_{n_r} \setminus \mathrm{St}(f^{-1}x_m, \mathfrak{B})).$$

Thus $fa_r = x_{n_r}$, while

$$\mathrm{St}(a_r, \mathfrak{B}) \cap f^{-1}x_m = \emptyset,$$

i.e.

$$x_m \notin V_{n_r} = f\,\mathrm{St}(a_{n_r}, \mathfrak{B}),$$

and for each r there are infinitely many such numbers m. This contradicts the preceding lemma: indeed, for some r and r_0 one has

$$V_{n_r} \supset \{x_{n_{r'}} \mid r' > r_0\},$$

and, consequently, it must also hold that

$$V_{n_r} \supset (x_i \mid i > i_0).$$

Now consider an arbitrary uniform cover $\mathcal{U}$ in the space M, and assume that $\mathfrak{B}$ is star-refined into it. Let

$$Z = \mathrm{St}(S_n, \mathfrak{B}) \in \mathfrak{Z},$$

where $n = n(\mathfrak{B})$ is the number found in the preceding paragraph. If $X \in \mathfrak{Z}$, then, for some m, $S_m \subset X$; consequently,

$$Z = Z(\mathcal{U}) \subset \mathrm{St}(\mathrm{St}(S_m, \mathfrak{B}), \mathfrak{B}) = \mathrm{St}(S_m, \mathfrak{B}^*) \subset \mathrm{St}(X, \mathcal{U}).$$

Thus the system $\mathfrak{Z}$ is indeed stable.

J. R. Isbell in ([3]) showed that every complete metric space satisfies the following ultracompleteness condition: an arbitrary stable regular* filter has nonempty intersection. Let

$$J = \bigcap \mathfrak{Z};$$

choose an arbitrary $u \in J$ and prove that $u = \lim x_i$. Indeed, let O be an open neighborhood of u. Put

$$G = f^{-1}u \cap J;$$

obviously,

$$G \subset f^{-1}O.$$

We shall now verify that

$$f^{-1}x_i \cap f^{-1}O \neq \emptyset$$

for almost all i: otherwise one would have

$$M \setminus f^{-1}O \supset \bigcup \{f^{-1}x_i \mid i \geq r\} = S_r, \qquad M \setminus G \supset M \setminus f^{-1}O,$$

and hence

$$M \setminus G \in \mathfrak{Z}.$$

But from this it should follow that $G \cap J = \emptyset$, i.e. $G = \emptyset$, which is absurd. Thus

$$f^{-1}O \cap f^{-1}x_i \neq \emptyset$$

and, consequently, $x_i \in O$ for almost all i.

We have proved that an arbitrary fundamental sequence

* The regularity condition for the system $\mathfrak{Z}$ means that for every $X \in \mathfrak{Z}$ there is a $Y \in \mathfrak{Z}$ with $X \supset Y$.

converge in P, i.e. P is sequentially complete. Since, by Theorem 1, this space is metrizable, it is complete.

Remark. In proving Theorem 2, we have established, in essence, the following more general assertion:

The image of an ultracomplete proximity space under countably compact proximity-open mappings is sequentially complete.

Noncompact equi-open mappings of metric spaces do not preserve the property of completeness, as the following simple example shows.

Example. Consider the set of points of the plane

$$\Psi = \bigcup \left\{ \left\{ \frac{1}{n} \right\} \times [n, \infty) \mid n = 1, 2, ... \right\}$$

with the induced proximity. Its projection onto the axis of abscissas

$$p : \Psi \to \left\{ \frac{1}{n} \mid n = 1, 2, ... \right\}$$

is a proximity-open mapping.

If, on the contrary, a mapping $\Phi : P \to Q$ of proximity spaces is "uniformly" open, i.e. if for every uniform cover $\mathfrak{P}$ there is a $\mathfrak{Q}$ such that

$$\Phi \ \mathrm{St}(x, \mathfrak{P}) \supset \mathrm{St}(\Phi x, \mathfrak{Q}),$$

then it is not difficult to prove the theorem on preservation of completeness (analogous to (4)) in full generality:

Theorem 3. *The image of an ultracomplete proximity space under a "uniformly" open mapping is ultracomplete.*

Proof. Let $\mathfrak{X}$ be a stable regular filter in Q. Consider the trace

$$\mathfrak{Y} = \{\Phi^{-1} X \mid X \in \mathfrak{X}\};$$

obviously, all the sets

$$Z_{\mathfrak{P}} = \bigcap \{\mathrm{St}(Y, \mathfrak{P}) \mid Y \in \mathfrak{Y}\}$$

are nonempty, since, by the properties of Φ and $\mathfrak{X}$, $\Phi Z_{\mathfrak{P}} \in \mathfrak{X}$. It is verified directly (independently of Φ) that the filter with trace

$$\mathfrak{Z} = \{Z_{\mathfrak{P}}\}$$

is stable. The set $\Phi(\cap\mathfrak{Z})$ is the limit of the filter $\mathfrak{X}$.

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Note: Figure translations are in progress. See original paper for figures.