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Abstract

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ON THE CARDINALITY OF BICOMPACTA WITH THE FIRST AXIOM OF COUNTABILITY

(Presented by Academician P. S. Aleksandrov on 23 V 1969)

In this paper we solve a problem posed almost half a century ago (in 1923) in the first publication of P. S. Aleksandrov and P. S. Uryson devoted to bicompact topological spaces (see [1−3]) (this problem belongs to Pavel Sergeevich Aleksandrov, as I know from him personally).

Theorem 1. *If a noncountable bicompactum satisfies the first axiom of countability, then its cardinality is equal to the cardinality of the continuum.*

Obviously, it is enough to prove that the cardinality of our bicompactum does not exceed the cardinality of the continuum.

This result will turn out to be a consequence of a general theorem. We shall prepare the formulation of the latter by a number of preliminary assertions.

Let $A \subset X$, where X is a topological space. By $[A]$ is denoted the closure of A in X, and by $|A|$ the cardinality of the set A. Let $x \in X$. By $\chi(x, X)$ is denoted the character of the point x in X, and moreover

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}.$$

By τ^+, where τ is a cardinal number, is denoted the first cardinal number greater than τ. By $\alpha + 1$ is denoted the transfinite ordinal immediately following the transfinite ordinal α. A space X is called sequential if the closed sets in X are exactly those sets which contain the limit of every convergent sequence of their elements.

Let T be some set and α some transfinite number. A tuple k of length α with values in T is any mapping of the set of all transfinite ordinals less than α into T (see [2]). The length of the tuple k is denoted by $l(k)$. Let k be a tuple of length α and let t be some element of the set T. Then (k, t) is the tuple which coincides, as a mapping, on all $\alpha' < \alpha$ with k and takes the value t at α. We say that the tuple k is an extension of the tuple k' if the mapping k' is a restriction of the mapping k, and in this case we write $k' \leqslant k$ (and $k' < k$, if, in addition, $k' \neq k$).

Lemma 1. *If X is a Hausdorff sequential space and $A \subset X$, $|A| \leqslant 2^\tau$, where $\tau \geqslant \aleph_0$, then also $|[A]| \leqslant 2^\tau$.*

Lemma 2. *If X is a sequential space and $A \subset X$, $x \in X$, $x \in [A]$, then there exists a set $A' \subset A$ for which $x \in [A']$ and $|A'| \leqslant \aleph_0$.*

Lemmas 1 and 2 easily follow from Theorem 3.15 of the article [(4)]—the second by induction.

Recall that a topological space X is called (τ, ∞)-compact if from every open covering of the space X one can select a covering of cardinality less than τ (see [(2,3)]). Recall also that the body of a covering is the union of all its elements.

Lemma 3. *Let X be a (τ^+, ∞)-compact T_1-space, where τ is a cardinal number, $\tau \geqslant \aleph_0$; let A be a closed subset of X; let $\chi(x, X) \leqslant 2^\tau$ for all $x \in A$, and let $|A| \leqslant 2^\tau$. Then $X \setminus A = \bigcup\{F_t : t \in T\}$, where $|T| \leqslant 2^\tau$ and each F_t is closed in X.*

For the proof of Lemma 3 it is enough to note that A has in X an external base $\mathcal{B}$ of cardinality $\leqslant 2^\tau$ (see [(5)]); therefore, by virtue of the closedness ···

A in X and (τ^+, ∞)-compactness of X, the bodies of covers of the set A of cardinality $\leq \tau$, composed of elements of $\mathcal{B}$, have intersection equal to the set A (we have also taken into account the equality $(2^\tau)^\tau = 2^\tau$ and the fact that X is a T_1-space).

Lemma 4. *Let τ be a cardinal number; $\tau \geq \aleph_0$; let X be a Hausdorff topological space and let $\chi(x, X) \leq \tau$ for all $x \in X$. Then, if $A \subset X$ and $|A| \leq 2^\tau$, then also $|[A]| \leq 2^\tau$.*

Proof. To each point $x \in [A]$ assign some set $A_x \subset A$ for which $|A_x| \leq \tau$ and $x \in [A_x]$. Let γ_x denote a base of cardinality $\leq \tau$ of the space X at the point x. Put

$$\lambda(x) = \{U \cap A_x : U \in \gamma_x\}, \quad x \in [A].$$

Then $|\lambda(x)| \leq \tau$ and $|U \cap A_x| \leq \tau$ for every $U \in \gamma_x$ and every $x \in [A]$, with $U \cap A_x \subset A$. Hence

$$|\{\lambda(x) : x \in [A]\}| \leq ((2^\tau)^\tau)^\tau = 2^\tau.$$

But $\lambda(x_1) \neq \lambda(x_2)$ whenever $x_1 \neq x_2$, since X satisfies the Hausdorff separation axiom. Thus $|[A]| \leq 2^\tau$.

Lemma 5. *If X is a (τ^+, ∞)-compact space, where $\tau \geq \aleph_0$, and τ' is a singular cardinal number greater than τ (in particular, $\tau' = \tau^+$ is suitable), then for a set $A \subset X$ of cardinality τ' in X there is a point of complete accumulation.*

Lemma 5 is an obvious and well-known assertion.

General theorem 2. *Let cardinal numbers τ, m and a topological space X be given, for which the following is known:*

(1) $\tau \geq \aleph_0$ and $m = 2^\tau$.

(2) X is a Hausdorff space.

(3) If $A \subset X$ and $|A| \leq m$, then also $|[A]| \leq m$.

(4) If A is closed in X and $|A| \leq m$, then

$$X \setminus A = \bigcup \{d_t(A) : t \in T\},$$

where $d_t(A)$ is closed in X for all $t \in T$, and T is some fixed set of cardinality m, the nature of whose elements is immaterial.

(5) If $A \subset X$ and $x \in X$, $x \in [A]$, then there exists $A

Note: Figure translations are in progress. See original paper for figures.

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