# THE METHOD OF EIGENFUNCTIONS IN AN ILL-POSED PROBLEM OF SYNTHESIS OF A LINEAR RADIATOR

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## Abstract

## Full Text

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# THE METHOD OF EIGENFUNCTIONS IN AN ILL-POSED PROBLEM OF SYNTHESIS OF A LINEAR RADIATOR

In paper ([1]) it was shown that the ill-posed problem of synthesis of a linear radiator with a prescribed radiation pattern $R(u)$ reduces to solving equation (1) for the amplitude-phase current distribution $J(x)$ in the antenna:

$$\alpha J(x) + 2\int_{-\sigma}^{\sigma} \frac{\sin u_0(x-\xi)}{x-\xi} J(\xi)\,d\xi - \int_{-u_0}^{u_0} R(u)\exp\{-iux\}\,du = 0, \qquad (1)$$

where $2\sigma$ is the electrical length of the radiator; $[-u_0, u_0]$ is the domain on which $R(u)$ is prescribed; $\alpha > 0$ is the regularization parameter. Methods for determining the optimal value of $\alpha$ are given in papers ([1,2]).

In paper ([1]) a method was proposed for solving equation (1) by means of grid functions ([3]), but this method determines the currents in discrete radiators located on $[-\sigma, \sigma]$, and consequently, instead of a continuous antenna, an array of radiators is synthesized. For a large antenna length $2\sigma$, such a method of solving equation (1) may lead to a considerable deviation of the grid function from the sought current distribution in the antenna. Therefore, at present a method of successive approximations for determining $J(x)$ is being developed; however, in a number of cases the process converges slowly and requires considerable computer time for its implementation. In papers ([1,4]) the possibility was indicated of determining (for any antenna length $2\sigma$) the amplitude-phase current distribution $J(x)$ by means of the theory of orthogonal functions. It is clear that the best implementation of this method from the computational point of view can be obtained if, as the system of orthogonal functions, one takes the system of eigenfunctions of the kernel of equation (1).

To determine the eigenfunctions and eigenvalues of the kernel of equation (1), we shall use the important fact established in ([5]):

$$\lambda_n S_{0,n}(c,y) = \int_{-1}^{1} \frac{\sin c(z-y)}{\pi(z-y)} S_{0,n}(c,z)\,dz; \qquad \lambda_n = \frac{2c}{\pi}\left[R_{0,n}^{(1)}(c,1)\right]^2, \qquad (2)$$

where $S_{0,n}(c,y)$ and $R^{(1)}_{0,n}(c,y)$ are, respectively, the angular and radial functions of the prolate spheroid, $c$ is a numerical parameter, and $\lambda_n$ are the eigenvalues of the operator (2). Making the change of variables $z = \xi/\sigma$, $y = x/\sigma$, $c = u_0\sigma$, and carrying out simple calculations, we obtain

$$\pi\lambda_n S_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right) = \int_{-\sigma}^{\sigma} \frac{\sin u_0(x-\xi)}{x-\xi} S_{0,n}\left(u_0\sigma, \frac{\xi}{\sigma}\right) d\xi. \tag{3}$$

Thus, the eigenfunctions of the kernel of equation (1) are the functions $S_{0,n}(u_0\sigma, x/\sigma)$, and the eigenvalues $\mu_n$ are as follows:

$$\mu_n = 2\pi\lambda_n = 4u_0\sigma\left[R^{(1)}_{0,n}(u_0\sigma, 1)\right]^2. \tag{4}$$

From comparison of (2) and (3), the physical meaning of the constant $c = u_0\sigma$ is clear: when considering the radiation pattern $R(u)$ on the physical inter-

of angles ($u_0 = 1$) $c = \sigma$, i.e., $c$ is one half of the electrical length of the antenna.

Thus, the amplitude-phase current distribution $J(x)$ (to within a factor $\alpha^{-1}$) has the form

$$J(x) = J_0(x) - \sum_n \frac{1}{\mu_n^{-1} + \alpha^{-1}} \left(J_0(x), S'_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right)\right) S_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right), \tag{5}$$

where

$$J_0(x) = \int_{-u_0}^{u_0} R(u)\exp\{-iux\}\,du,$$

and $\left(J_0(x), S'_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right)\right)$ is the scalar product of functions in the space $L_2$. Let us examine formula (5) in more detail. It is easy to show that, for any $n$,

$$\left|\left(J_0(x), S_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right)\right)\right| < 4u_0\sigma,$$

and, moreover, it decreases rapidly as $n$ increases. In addition, it follows from (4) and from the results of [6] that, for $n > 2u_0\sigma/\pi + 1$, $\mu_n < 0.02$, and the $\mu_n$ decrease by more than an order of magnitude in passing to $\mu_{n+1}$. Therefore the series (5) must converge rapidly for almost all $\alpha$. Incidentally, we note that choosing $N \le 2u_0\sigma/\pi + 1$ eliminates the phenomenon of superdirectivity [7]. It is easy to show, using [6], that for $N \le [u_0\sigma/2 + 1]$ and $u_0\sigma \ge 25$, with an accuracy no worse than 0.3%, one may take $\mu_n = 2\pi$ ($n = 1, 2, \dots, N$).

For practical computations, an analytic or tabular specification of the functions $S_{0,n}(c, x')$ is necessary. At present the functions $S_{0,n}(c, x')$ have been tabulated only for $c \leq 8$, and they can be used for calculating antennas of electrical length $2\sigma \leq 1.6$. However, judging from the direction of works [6, 8], the functions $S_{0,n}(c, x')$ will soon be tabulated for $8 < c < 100$. The basis for this is the asymptotic formulas for $S_{0,n}(c, x')$ obtained in [8]. Taking into account the results of this work and relation (3), the functions $S_{0,n}(u_0\sigma, x/\sigma)$ have the form ($n \leq u_0\sigma$; in practice these formulas may be used for $n \leq u_0\sigma/2$):

$$S_{0,n}\left(u_0\sigma, \frac{x}{\sigma}\right) = \begin{cases} S_{0,n}^{(1)}(x), & 0 \leq x \leq u_0^{-1}, \\ k_2 S_{0,n}^{(2)}(x), & u_0^{-1} \leq x \leq \sigma - u_0^{-1}, \\ k_3 S_{0,n}^{(3)}(x), & \sigma - u_0^{-1} \leq x \leq \sigma; \end{cases} \tag{6}$$

$$k_2 = e^{-u_0\sigma}(u_0\sigma)^{n/2} 2^{(3n+1)/2} P(u_0\sigma); \qquad k_3 = k_2(2\pi u_0\sigma)^{1/2}; \tag{7}$$

$$S_{0,n}^{(1)}(x) = D_n\left(x\sqrt{\frac{2u_0}{\sigma}}\right) + \sum_{j=1}\left(\frac{1}{2u_0\sigma}\right)^j \sum_{k=-2j}^{2j} A_k^j D_{n+2k}\left(x\sqrt{\frac{2u_0}{\sigma}}\right);$$

$$S_{0,n}^{(2)}(x) = \frac{e^{u_0\sigma t}(1-t)^{n/2}}{t^{1/2}(1+t)^{(n+1)/2}}\left[1 + \sum_{j=1}\left(\frac{1}{2u_0\sigma}\right)^j \sum_{k=1}^{j}\left(\frac{D_k^j}{t^k} + \frac{E_k^j}{(1+t)^k} + \frac{F_k^j}{(1-t)^k}\right)\right];$$

$$S_{0,n}^{(3)}(x) = I_0(u_0\sigma t) + \sum_{j=1}\left(\frac{1}{2u_0\sigma}\right)^j \sum_{k=1}^{j} B_k^j [u_0\sigma t]^k I_k(u_0\sigma t),$$

$$t = \sqrt{1-(x/\sigma)^2};$$

$A_k^j, D_k^j, E_k^j, F_k^j, B_k^j, P(u_0\sigma)$ are numerical coefficients of polynomial form depending only on $n$; their expressions are given in [8]; $D_n(x\sqrt{2u_0/\sigma})$ is a Weber function; $I_k(u_0\sigma t)$ is a modified Bessel function.

From examination of formulas (6)–(8) it is clear that, for computations with a large electrical length of the antenna, it is sufficient to restrict oneself to the value $j = 2, 3$. This substantially simplifies the computation of the function $S_{0,n}(u_0\sigma, x/\sigma)$.

Finally, we note that by the methods of ill-posed problems one can also solve the classical synthesis problem for a given coefficien-

of superdirectivity $Q$ (7). In this case it is not difficult to show, using the results of work (1), that it is necessary to solve the equation

$$aJ(x) + 2(1-\alpha Q)\int_{-\sigma}^{\sigma} \frac{\sin(x-\xi)}{x-\xi} J(\xi)\, d\xi - \int_{-1}^{1} R(u) \exp\{-iux\}\, du = 0,$$

and $J(x)$ can be found by means of the method described above.

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## CITED LITERATURE

1. L. D. Bakhrakh, S. D. Kremenetskii, DAN, 178, No. 4, 825 (1968).
2. S. D. Kremenetskii, M. E. Zhukovina, *Problems of Radio Electronics*, General-Technical Series, issue 19, 26 (1968).
3. A. N. Tikhonov, DAN, 151, No. 3, 501 (1963).
4. V. I. Popovkin, V. I. Elumeev, *Radio Engineering and Electronics*, 13, No. 5, 784 (1968).
5. D. Slepian, H. O. Pollak, Bell Syst. Techn. J., 40, 43 (1961).
6. D. Slepian, E. Sonnenblick, ibid., 44, No. 8, 1745 (1965).
7. D. R. Rhodes, IEEE Trans. on Antennas and Propagation, AP-11, 440 (1963).
8. D. Slepian, J. Math. and Phys., 44, No. 2, 100 (1965).

*Note: Figure translations are in progress. See original paper for figures.*