PHYSICS

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1969

SovietRxiv

Abstract

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A ONE-DIMENSIONAL NONLINEAR MODEL OF THE INSTABILITY OF AN ANISOTROPIC PLASMA

1. At present there exists neither a sufficiently reliable analytical theory of the nonlinear stage of plasma instability nor adequate numerical methods. The exceptions are one-dimensional models, in which it is possible not only to trace the development of instabilities by means of computers, but even to find certain classes of analytical solutions. One-dimensional models of beam electrostatic instabilities have been studied in the greatest detail ([1,2]).

We shall consider another case, namely a model of an anisotropic plasma that is unstable with respect to the growth of Alfvén waves under the condition $p_\parallel > p_\perp + H_0^2/4\pi$, where $p_\parallel (p_\perp)$ is the plasma pressure along (across) the unperturbed magnetic field H_0 (firehose instability). Interest in this case is due to the fact that the turbulence developing as a result of the firehose instability can apparently play an important role in processes occurring in the solar wind ([3]). This model is also of methodological interest, since even in a collisionless plasma Alfvén turbulence resembles hydrodynamic turbulence, and it can be considered on the basis of hydrodynamic-type equations ([3]).

2. As the initial equations we take the magnetohydrodynamic equations in which the finite ion Larmor radius is taken into account ([4]), and which are a generalization of the model of Chew, Goldberger, and Low ([5]). In the one-dimensional case, when all quantities depend only on z and t (the direction of the unperturbed magnetic field H_0 is chosen as the axis), these equations have the form:

$$
\begin{gathered}
\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial z}(\rho w)=0, \\
\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial z}\left\{\rho u w+(p_{\|}-p_{\perp}) \frac{H_0 H_x}{H^2}-\frac{H_0 H_x}{4 \pi}-\right. \\
\left.-\frac{1}{\omega_H}\left[(p_{\|}-p_{\perp}) \frac{H_0 H_y}{H^2} \frac{\partial w}{\partial z}\left(p_{\perp}+\frac{p_{\|}-p_{\perp}}{H^2} H_0^2\right) \frac{\partial v}{\partial z}\right]\right\}=0, \\
\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial z}\left\{\rho v w+(p_{\|}-p_{\perp}) \frac{H_0 H_y}{H^2}-\frac{H_0 H_y}{4 \pi}+\right. \\
\left.+\frac{1}{\omega_H}\left[(p_{\|}-p_{\perp}) \frac{H_0 H_x}{H^2} \frac{\partial w}{\partial z}+\left(p_{\perp}+\frac{p_{\|}-p_{\perp}}{H^2} H_0^2\right) \frac{\partial u}{\partial z}\right]\right\}=0, \\
\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial z}\left(\rho w^2+p_{\perp}+\frac{p_{\|}-p_{\perp}}{H^2} H_0^2+\frac{H_x^2+H_y^2}{8 \pi}\right)=0, \\
\frac{\partial H_x}{\partial t}+\frac{\partial}{\partial z}(w H_x-u H_0)=0, \\
\frac{\partial H_y}{\partial t}+\frac{\partial}{\partial z}(w H_y-v H_0)=0, \\
\frac{\partial}{\partial t}\left(\frac{p_{\|} H^2}{\rho^3}\right)+w \frac{\partial}{\partial z}\left(\frac{p_{\|} H^2}{\rho^3}\right)=0, \\
\frac{\partial}{\partial t}\left(\frac{p_{\perp}}{\rho H}\right)+w \frac{\partial}{\partial z}\left(\frac{p_{\perp}}{\rho H}\right)=0 .
\end{gathered} \tag{1}
$$

Here $H^2=H_0^2+H_x^2+H_y^2$, ω_H is the ion cyclotron frequency, $u=v_x$, $v=v_y$, $w=v_z$.

For Alfvén waves propagating along the constant magnetic field H_0, the dispersion law has the form

$$
\omega=\omega_0+i \gamma, \qquad \omega_0=\frac{1}{2} \omega_H k^2 R^2,
$$

$$
\gamma=\omega_H k R \sqrt{\frac{1}{p_{\|}}\left(p_{\|}-p_{\perp}-\frac{H_0^2}{4 \pi}\right)-\frac{k^2 R^2}{4}}, \tag{2}
$$

where

$$
R=\frac{1}{\omega_H} \sqrt{\frac{p_{\|}}{\rho}}
$$

is the ion Larmor radius.

From expression (2) for the growth increment of small perturbations it follows that the short-wave harmonics, when the finite ion Larmor radius is taken into account, are stabilized. The hose instability thus develops when the conditions

Fig. 1

$$p_{\parallel} > p_{\perp} + \frac{H_0^2}{4\pi}, \qquad kR < 2\sqrt{\frac{1}{p_{\parallel}}\left(p_{\parallel} - p_{\perp} - \frac{H_0^2}{4\pi}\right)}. \tag{3}$$

3. The system of equations (1) admits an analytic solution for arbitrary amplitudes in the form

$$H_x = H_0 B(t)\sin(kz + \varphi(t)), \qquad H_y = H_0 B(t)\cos(kz + \varphi(t)), \tag{4}$$

where $B(t)$, $\varphi(t)$ are functions of time.

In this case the magnetic pressure does not depend on the spatial coordinate; therefore the longitudinal motion is not excited if it was absent at the initial instant, and the pressures $p_{\parallel}, p_{\perp}$ also prove to depend only on time, i.e.

$$p_{\parallel} = \frac{p_{\parallel}^0}{1 + B^2(t)}, \qquad p_{\perp} = p_{\perp}^0\sqrt{1 + B^2(t)}, \tag{5}$$

where $p_{\parallel}^0, p_{\perp}^0$ are the initial pressures.

Substituting expressions (4) into the original system (1), we obtain the equations:

$$\ddot{B} - B\dot{\varphi}^2 - \frac{k^2}{\rho_0\omega_H}\left(p_{\perp} + \frac{p_{\parallel} - p_{\perp}}{1 + B^2}\right)B\dot{\varphi} = \frac{k^2}{\rho_0}\left(\frac{p_{\parallel} - p_{\perp}}{1 + B^2} - \frac{H_0^2}{4\pi}\right)B,$$

$$B\ddot{\varphi} + 2\dot{B}\dot{\varphi} = -\frac{k^2}{\rho_0\omega_H}\left(p_{\perp} + \frac{p_{\parallel} - p_{\perp}}{1 + B^2}\right)\dot{B}. \tag{6}$$

Integrating system (6) gives

$$\dot{\varphi} = -\frac{\omega_0}{B^2}\left\{1 + \frac{4}{3}\alpha + \frac{2}{3}\alpha(B^2 - 2)\sqrt{1 + B^2} - \frac{1}{1 + B^2}\right\}, \qquad \alpha = \frac{p_{\perp}^0}{p_{\parallel}^0}, \tag{7}$$

$$\dot{B}^2 + U(B) = E = \text{const},$$

where the "potential energy" is equal to

$$U(B) = \frac{k^2 p_\parallel^0}{\rho^0}\left(\frac{1}{1+B^2} + 2\alpha\sqrt{1+B^2} + \frac{H_0^2}{4\pi p_\parallel^0}B^2\right) + \\ + \omega_0^2\left\{\frac{\alpha^2}{9}\left(\frac{32}{B^2} - 12B^2 + 4B^4 - 32\frac{\sqrt{1+B^2}}{B^2} + 16\sqrt{1+B^2}\right) + \\ + \frac{4}{3}\alpha\left(\sqrt{1+B^2} - \frac{2B^2}{1+B^2} - \frac{3}{\sqrt{1+B^2}}\right) + \frac{1+2B^2}{1+B^2} - \frac{B^4}{(1+B^2)^2}\right\}. \quad (8)$$

The "total energy" E is determined by the level of the initial perturbation. Thus, the system of equations (1) has an analytic solution describing a nonlinear monochromatic wave. The amplitude of the wave

Fig. 2

changes periodically with time; the character of these oscillations is determined by the form of the potential well $U(B)$ and by the magnitude of the initial perturbation.

Let us consider the limiting case of sufficiently small magnetic fields, expanding the function U in a series in powers of B^2 up to and including terms in B^4. As a result we obtain

$$U(B) = B^2\left\{-\gamma^2 + \omega_H^2 k^2 R^2\left[1 - \frac{\alpha}{4} - \frac{1}{2}\left(1 - \frac{\alpha}{2}\right)k^2R^2\right]B^2\right\}, \quad (9)$$

and the maximum value of the magnetic-field amplitude is approximately equal to

$$B_{\max} \approx \frac{\gamma}{\sqrt{1 - \alpha/4 - {}^1/_2(1-\alpha/2)\,k^2R^2}\,\omega_H kR}. \quad (10)$$

Consequently, for a small degree of plasma anisotropy the amplitude of the magnetic field in a monochromatic wave with circular polarization is directly proportional to the increment of the fire-hose instability.

4. As was shown above, for an initial perturbation in the form of a monochromatic wave with circular polarization $H_x(z,0) = B^{(0)}\sin kz$, $H_y(z,0) = B^{(0)}\cos kz$, there is an analytic solution of the problem of nonlinear fire-hose instability. In the case of perturbations of arbitrary type there is no such solution, and in order to investigate the nonlinear stage of the instability it is necessary to carry out a numerical solution. The system of equations (1) was solved on a computer by passing to an explicit finite-difference scheme with periodic boundary conditions. As initial conditions, one or several sinusoidal waves with different phase shifts were chosen.

Fig. 3

Figure 1: Fig. 3

Fig. 3

Figure 1 gives the time dependence of the averaged magnetic pressure $p_m = \langle H_\perp^2/8\pi\rangle$ for the case $p_\parallel^0 = 30H_0^2/8\pi$, $p_\perp^0 = 6.6H_0^2/8\pi$ and an initial perturbation in the form of a monochromatic wave with circular polarization. The magnetic pressure is averaged over the spatial computational interval. At small times the dependence $p_m(t)$ agrees quite well with the analytic solution (i.e., the magnetic field reaches a maximum, and then returns to small values). With time, however, the numerical solution departs from the analytic one and acquires other features: regular oscillations of the magnetic pressure disappear and the spatial distribution of the magnetic field becomes stochastic (Fig. 2). The changes noted are a consequence of the fact that uncontrolled computational noise (the discreteness of the finite-difference approximation, rounding errors, etc.) grows in accordance with the plasma instability described by the system of equations (1). We note that such a situation is close to the real physical one, when perturbations are likewise uncontrolled. If two sinusoidal waves with different wavelengths are taken as the initial perturbation, then regular oscillations and return are absent.

As the calculation shows, after a time of the order of 50 inverse instability increments the averaged magnetic pressure reaches a quasistationary level, whose magnitude increases with increasing degree of plasma anisotropy. The mean square of the turbulent magnetic field as a function of the ratio $p_\parallel^0/p_\perp^0$ (of the longitudinal pressure to the transverse pressure) is shown in Fig. 3. The magnitude of this quasistationary magnetic pressure p_m, for small plasma anisotropy ($p_\parallel^0/p_\perp^0 \simeq 1.25 \div 1.8$), increases approximately linearly with the increase of the ratio $p_\parallel^0/p_\perp^0$; at $p_\parallel^0/p_\perp^0 \simeq 8$, p_m reaches the value $\simeq 2H_0^2/8\pi$, and then the growth of the turbulent magnetic pressure is sharply slowed (an almost plateau is formed), and at $p_\parallel^0/p_\perp^0 = 50$, $p_m \simeq 4H_0^2/8\pi$.

Received
14 X 1968

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Note: Figure translations are in progress. See original paper for figures.