THE PRINCIPLE OF COMPARISON OF INDICES AND SEPARATION LAWS IN CERTAIN CLASSES OF SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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THE PRINCIPLE OF COMPARISON OF INDICES AND SEPARATION LAWS IN CERTAIN CLASSES OF SETS OF TOPOLOGICAL SPACES OF WEIGHT τ

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A. A. Lyapunov ([1]) obtained a principle of comparison of indices for stabilizing iterations of set-theoretic (s.-t.) transformations as applied to an arbitrary space of indices. Hence there immediately follows the principle of comparison of indices for Z-operations with chain depth ω. Pseudomonotone and contracting s.-t. transformations will be called regular.

Theorem 1. If $Z_{\{N_i\}}$ and $Z_{\{M_j\}}$ are regular s.-t. transformations, $(E_i)_i$ and $(H_j)_j$ are arbitrary families of sets of the basic space $\mathfrak{X}$,

$$U_{ij} = E_i \cup CH_j, \qquad \Psi_{Q_{ij}}\{U_{ij}\} = U_{i_1} \cap \Psi_{j', M_j^c}\{\Psi_{i', N_i}\{U_{i'j'}\}\},$$

then

$$Z_{\{Q_{ij}\}}\{U_{ij}\} = [Z_{\{N_i\}} \operatorname{Ind}(x \mid \{E_i\}) \geq Z_{\{M_j\}} \operatorname{Ind}(x \mid \{H_j\})],$$

in particular,

$$T_{\{Q_{ij}\}}\{U_{ij}\} = [T_{\{N_i\}} \operatorname{Ind}(x \mid \{E_i\}) \geq T_{\{M_j\}} \operatorname{Ind}(x \mid \{H_j\})].$$

Let the space of indices I have cardinality $\tau = \aleph_\nu$, where τ is a strongly inaccessible cardinal number ([2]). In view of the equivalence of Q- and R-operations to Z- and T-operations without hooks, the principle of comparison of indices also holds for Q- and R-operations.

Let

$$\mathfrak{M}_1 = (N_i) \cup (N_{(i_\alpha)_\gamma}) \quad \text{and} \quad \mathfrak{M}_2 = (M_j) \cup (M_{(j_\alpha)_\gamma})$$

be bases of $(\upsilon)T$-operations, $\upsilon = \omega \cdot \lambda \leq \omega_\nu$. If we put

$$U_{ij} = E_i \cup CH_j, \qquad \Phi_{Q_{ij}}\{U_{ij}\} = \Phi_{j',\,M_j^c}\{\Phi_{i',\,N_i}\{U_{i'j'}\}\},$$

$$\Phi_{L_{ij}^{\upsilon'}}\{U_{ij}\} = \Phi_{j',\,M_j^{\upsilon' c}}\{\Phi_{i',\,N_i^{\upsilon'}}\{U_{i'j'}\}\}, \qquad \upsilon' = \omega \cdot \lambda' < \upsilon,$$

then the R-iteration of s.-t. transformations with variable bases $(Q_{ij})_{i,j}$, $(L_{ij}^{\upsilon'})_{i,j,\upsilon}$, performed over the family of sets $(U_{ij})_{i,j}$, gives

$$U_{ij}^\alpha = \bigcap_{\beta<\alpha} (E_i^\beta \cup CH_j^\beta)$$

for any $\alpha < \xi(\upsilon)$. Hence we obtain:

Theorem 2. If

$$\mathfrak{M}_1 = (N_j) \cup (N_{(j_\alpha)_\gamma}), \qquad \mathfrak{M}_2 = (M_j) \cup (M_{(j_\alpha)_\gamma})$$

are families of bases of $\Delta\Sigma$-operations, where $i, j \in I$, $(j_\alpha)_\gamma$ are all possible coordinated sequences of indices of type $\gamma < \upsilon = \omega \cdot \lambda \leq \omega_\nu$, $(E_i)_i$, $(H_j)_j$ are arbitrary families of sets of the basic space,

$$U_{ij} = E_i \cup CH_j,$$

$$\Phi_{Q_{ij}}\{U_{i'j'}\} = \Phi_{j',\,M_j^c}\{\Phi_{i',\,N_i}\{U_{i'j'}\}\}, \qquad \Phi_{L_{ij}^{\upsilon'}}\{U_{ij}\} = \Phi_{j',\,M_j^{\upsilon' c}}\{\Phi_{i',\,N_i^{\upsilon'}}\{U_{i'j'}\}\}$$

with $\upsilon' = \omega \cdot \lambda' < \upsilon$,

$$\Omega = (Q_{ij}) \cup (L_{ij}^{\upsilon'}),$$

then

$$(\upsilon)T_{\{ij\}\Omega}\{U_{ij}\} = U_\omega^{\xi(\upsilon)} = \big[(\upsilon)T_{\mathfrak{M}_1}\,\mathrm{Ind}(x \mid \{E_i\}) \geq (\upsilon)T_{\mathfrak{M}_2}\,\mathrm{Ind}(x \mid \{H_j\})\big].$$

Let M be a base of the $\Delta\Sigma$-operation satisfying the following conditions:

1°. $(\upsilon)T_M \succ \bigcup_\tau$, $(\upsilon)T_M \succ \bigcap_\tau$, $(\upsilon)T_M \succ \Phi_{M^c}$ and $(\upsilon)T_M \succ \Phi_{M^{\omega\cdot\lambda'}}$, $(\upsilon)T_M \succ \Phi_{M^{\omega\cdot\lambda'};c}$; $\omega \cdot \lambda' < \upsilon$.

2°. $(\Phi_M, d) \succ \Phi_M$, $(\Phi_{M^c}, d) \succ \Phi_{M^c}$.

Denote by $(\upsilon)\mathfrak{B}_M$ the class of $\Delta\Sigma$-operations constructed as follows: 1) the trivial operations, the operations $\bigcap_\tau$, $\bigcup_\tau$, Φ_M, Φ_{M^c}, $\Phi_{M^{\omega\cdot\lambda'}}$, $\Phi_{M^{\omega\cdot\lambda';c}}$ with $\omega\cdot\lambda' < \upsilon$ belong to the class $(\upsilon)\mathfrak{B}_M$; 2) if $\Phi_L \in (\upsilon)\mathfrak{B}_M$, $\Phi_{M_i} \in (\upsilon)\mathfrak{B}_M$ for $i \in I$, then $\Phi_L\{\Phi_{M_i}\}_i \in (\upsilon)\mathfrak{B}_M$; 3) the class $(\upsilon)\mathfrak{B}_M$ is invariant with respect to shifts of bases; 4) $(\upsilon)\mathfrak{B}_M$ is the smallest class of sets satisfying conditions 1)–3). The operations belonging to the class $(\upsilon)\mathfrak{B}_M$ are no stronger than operations of type $(\upsilon)T_M$. Denote by $(\upsilon)T\mathfrak{B}_M$ the class of υ-extensions of the $\Delta\Sigma$-operations of the class $(\upsilon)\mathfrak{B}_M$. The operations belonging to the class $(\upsilon)T\mathfrak{B}_M$ are no stronger than operations of type $(\upsilon)T_{T_M}$, and hence also than operations of type $(\upsilon)T_M$.

Theorem 3. *If $\mathfrak{M} = (W_i) \cup (W_{(i_\alpha)_\gamma})$, $\mathfrak{E} = (S_i) \cup (S_{(i_\alpha)_\gamma})$ are bases of $\Delta\Sigma$-operations of the class $(\omega_\upsilon)\mathfrak{B}_M$, where $i \in I$, $(i_\alpha)_\gamma$ are all possible compatible sequences of indices of type $\gamma < \omega_\upsilon$, $(E_{ij})_{i,j}$, $(\mathcal{E}_{ij})_{i,j}$ are arbitrary families of sets of the class $K^c \supseteq \emptyset$, $\mathfrak{M}_x$, such that the class $K^c \subset \mathfrak{M}_{xy_0\cdots y}$ is projective, for any $i \in I$, with respect to the class $K^c \subset \mathfrak{M}_x$ with base $L(y_0,\dots,y_i) = \{\eta\}$, where $\eta \in N$, $\Phi_N \equiv (\omega_\upsilon)T_{\{M\}}$, and $K \cup K^c \subset \Phi_Q(K^c)$, where $\Phi_Q \prec (\omega_\upsilon)T_M$, $(\omega_\upsilon)P_{\hat{N};M_T} \prec (\omega_\upsilon)T_{\{M\}}$,*

$$N = \bigcup_{(y_0,\dots,y_i)\in\mathfrak{M}_{y_0\cdots y_i}} L(y_0,\dots,y_i),$$

$$\beta_1(x) = (\omega_\upsilon)P^c_{\hat{N};\mathfrak{M}}\operatorname{Ind}(x/\{E_{ij}\}), \quad \beta_2(x) = (\omega_\upsilon)P^{\hat{}}_{\hat{N};\mathfrak{E}}\operatorname{Ind}(x/\{\mathcal{E}_{ij}\}),$$

then

$$[\beta_1(x) \geqslant \beta_2(x)] \in (\omega_\upsilon)P^c_{\hat{N}M}(K^c).$$

For $\tau = \aleph_0$, we have P. S. Novikov' s principle of comparison of indices ([4]).

Theorem 4. *Let $(E_i)_i$, $(\mathcal{E}_i)_i$ be arbitrary families of sets of class K, invariant with respect to the operations of complementation and union of finite families of sets, and suppose $E_i \supset E_{i+1}$, $\mathcal{E}_i \supset \mathcal{E}_{i+1}$ for $i \in I$. Then, for an arbitrary cardinal number τ,*

$$\left[\bigcap_\tau \operatorname{Ind}(x/\{E_i\}) > \bigcap_\tau \operatorname{Ind}(x/\{\mathcal{E}_i\})\right] \in \Phi_{\bigcup_\tau}(K).$$

Consider some rearrangements of the operations $Z_{\{N_j\}}\{E_i\}$, leading to a change in the indices of the given operation. Construct a quasi-shift of bases $f(j) = j+n$, i.e. put

$$M_{j+n} = \bigcup_{\eta\in N_j} \bigcap_{i\in f(\eta)} E^i \cap \bigcap_{i\in f(I)\setminus f(\eta)} CE^i.$$

In addition, let $M_\gamma = E^{\gamma,\gamma+1}$, $M_{\gamma+1} = E^{\gamma+1,\gamma+2}$, ..., $M_{\gamma+n-1} = E^{\gamma+n-1,\gamma+n}$. Put $H_\gamma = H_{\gamma+1} = \cdots = H_{\gamma+n} = E_\gamma$, $H_{i+n} = E_i$, if $i \neq \gamma$, where $\gamma < \omega_v$ is a limit ordinal. Then $\Psi_{M_{j+n}}\{H_i\} = \Psi_{N_j}\{E_i\}$ for $j \in I$, $Z_{\{M_j\}}\{H_i\} = Z_{\{N_j\}}\{E_i\}$, while at the same time

$$Z_{\{M_j\}} \operatorname{Ind}(x/\{H_i\}) = Z_{\{N_j\}} \operatorname{Ind}(x/\{E_i\}) + n,$$

if

$$x \in CE_0^{\omega_v+1} = CZ_{\{N_j\}}\{E_i\} \left(\exists \equiv \Phi I = E[\eta \subset I]\right).$$

If we construct a new family of sets $H_{ni} = H_{ni+1} = \cdots = H_{ni+n-1} = E_i$ for $i \in I$ and a new family of bases $M_{ni+k} = E^{ni+k,\, ni+k+1}$ for $k \leq n-2$, putting $f(i) = ni$, then we obtain

$$M_{ni+n-1} = \bigcup_{\eta \in N_j;\ i \in f(\eta) \cup \{ni+n-1\}} \bigcap_{i \in f(I) \setminus f(\eta)} E^i \cap CE^i.$$

Then

$$\Psi_{M_{ni+n-1}}\{H_i\} = H_{ni+(n-1)} \cap \bigcap_i \Psi_{N_i}\{H_{ni}\},$$

$$\Psi_{M_{ni+k}}\{H_i\} = H_{ni+k} \cap H_{ni+k+1}, \qquad k \leq n-2,$$

$$Z_{\{M_j\}}\{H_j\} = Z_{\{N_j\}}\{E_i\}.$$

At the same time, if

$$\beta(x) = Z_{\{N_j\}} \operatorname{Ind}(x/\{E_i\}), \quad \beta_1(x) =$$

$$= Z_{\{M_i\}} \operatorname{Ind}(x \mid \{H_i\}),$$

and $\beta(x) = \alpha^* + k$, where α^* is a limit ordinal, $k < \omega$, then

$$\beta_1(x) = \alpha^* + nk \quad \text{for } x \in CZ_{\{N_j\}}\{E_i\}.$$

Analogously one can carry out rearrangements of T-operations with chains of arbitrary depth, using the shift of bases $f(i) = i + n$ when the index is increased by n units, and $f(i) = ni$ when an n-fold index is obtained.

Let $\Lambda = (\beta(x))$ be a class of transfinite indices. Denote by $K(\Lambda)$ the class of all sets representable in the form $[\beta(x) = \omega_{\nu+1}]$ $([\beta(x) = \zeta(\omega_\nu)])$. The class Λ is said to be regular if $[\beta_1(x) \geq \beta_2(x)] \in K(\Lambda)$ for all functions $\beta_1(x), \beta_2(x) \in \Lambda$, and completely regular if, in addition,

$$(\forall E_1, E_2 \in K(\Lambda))(\exists \beta_1(x), \beta_2(x) \in \Lambda)\,[x \in E_1 \cap E_2 \equiv \beta_1(x) = \beta_2(x)].$$

From the principles of comparison of indices and rearrangement of operations leading to a change of indices, there follows the complete regularity of the following classes of transfinite indices:

1) The class of external indices of the operation $\bigcap_\tau$ with respect to the class of sets $\Phi_\eta(K)$, if the class K is invariant with respect to the operations of complementation and union of sets in number $< \tau$.
2) The class of external indices (ω_ν) of a T-operation (of T-operations) with bases of operations of the class $(\omega_\nu)\mathfrak{B}_N(\mathfrak{B}_N)$, if $K \supset \mathfrak{A}$ is invariant with respect to the operations of complementation and union of sets of any finite family.
3) The class of external indices of the operation $(\omega_\nu)P^c_{\tilde{N}\mathfrak{M}}$, where the bases of the family
$$\mathfrak{M} = (M_i) \cup (M_{(i\alpha)\gamma})$$
belong to operations of the class $(\omega_\nu)\mathfrak{B}_M$, if the class $K \supset \emptyset, \mathfrak{A}$ is such that
$$K^c \in \mathfrak{PM}_{xy_0\cdots y_i}$$
for $i \in I$ is projective with respect to the class $K^c \in \mathfrak{PM}_x$ with base
$$L(y_0, \dots, y_i) = \{\eta\},$$
where $\eta \in N$, $\Phi_N = (\omega_\nu)T_{\{M\}}$, and, moreover,
$$N = \bigcup_{(y_0,\dots,y_i)\in\mathfrak{A}_{y_0\cdots y_i}} L(y_0, \dots, y_i), \qquad K \cup K^c \subset \Phi_Q(K^c), \quad \Phi_Q \prec (\omega_\nu)T_M,$$
$$(\omega_\nu)P_{\tilde{N}M_T} \prec (\omega_\nu)T_M, \qquad P_{\tilde{N}M_T} \equiv \Phi_{\tilde{N}\tilde{L}}, \qquad \Phi_L = (\omega_\nu)T_{\{M\}}.$$

At the basis of the theory of multiple separability, together with the principle of comparison of indices, lies the following fundamental lemma:

Lemma 1. Let $\Lambda = (\beta_i(x))$ be a family of transfinite functions, Φ_N some $\Delta\Sigma$-operation, and

$$E_i = [\beta_i(x) = \xi(\nu)],$$

where $\nu = \omega \cdot \lambda \leq \omega_\nu$,

$$Q_{ij} = [\beta_i(x) \leq \beta_j(x)], \qquad H_i = C\Phi_{N^i}\{Q_{ij}\}.$$

Then

$$H_i \supset E_i \setminus \Phi_{N^i}\{E_j\}, \qquad \Phi_N\{H_i\} = \emptyset.$$

For $\tau = \aleph_0$ the lemma was proved by A. A. Lyapunov ([3]). For general t.-m. operations there is the following more restricted lemma:

Lemma 2. Let $\Lambda = (\beta_i(x))$ be a family of transfinite indices, Ψ_N some t.-m. operation,

$$E_i = [\beta_i(x) = \xi(\nu)], \qquad Q_{ij} = [\beta_i(x) \le \beta_j(x)], \qquad H_i = C\Psi_{N^i}\{Q_{ij}\}.$$

Then

$$H_i \supset E_i \setminus \Psi_{N^i}\{E_j\}.$$

Hence, on the basis of the general theory of separability developed by A. A. Lyapunov ([3,5]), we obtain the following assertions:

1. For the classes of sets F, $F^{\alpha+2}$ of τ-spaces J^ω, D^ω for $\alpha < \omega_{\nu+1}$, $F^{\alpha+2}$ for $\alpha \ge \omega$, $F^{\alpha+3}$ for $\alpha < \omega$ of the spaces J^τ, D^τ, the first and second separability theorems hold, as do the first and second theorems on multiple separability with respect to the operation $\bigcap_{\tau'}$ for $\tau' \le \tau$; the third separability theorem and the third theorem on multiple separability with respect to the operations $\bigcap_\tau$, lim do not hold. For the class F of the spaces J^τ, D^τ for $\tau > \aleph_0$, the first separability theorem holds, as does the first theorem on multiple separability with respect to the operation of finite intersection, and the second separability theorem and the second theorem on multiple separability with respect to ···

with respect to the same operation. For the classes G, $G^{\alpha+2}$ of τ-spaces $J^{\omega_\nu}, D^{\omega_\nu}$ for $\alpha < \omega_{\nu+1}$, the class $G^{\alpha+2}$ for $\alpha \ge \omega$, and $G^{\alpha+3}$ for $\alpha < \omega$ of the spaces J^τ, D^τ, the third separation theorem and the third theorem on multiple separation with respect to the operation of finite intersection hold; the first and second separation theorems with respect to the operations $\cap$, lim do not hold. For the class G of the spaces J^τ, D^τ, when $\tau < \aleph_0$, the second and third separation theorems and the third theorem on multiple separation with respect to the operation of finite intersection do not hold.

2. For the class of $(\omega_\nu)A$-sets obtained by the A-operation with complete depth of chains ω_ν, when $\tau > \aleph_0$, the separation laws in the spaces under study $J^{\omega_\nu}, D^{\omega_\nu}, J^\tau, D^\tau$ are analogous to the separation laws for the class of A-sets of the Baire space J. The same holds for higher classes of $(\omega_\nu)R$-sets.

3. For the second class of projective sets of the space J^{ω_ν}, when $\tau > \aleph_0$, we obtain separation laws converse to the separation laws in the first class of projective sets of this space (in the class of $(\omega_\nu)A$-sets).

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Note: Figure translations are in progress. See original paper for figures.