# ON EQUATIONS ARISING IN THE SUMMATION OF PERTURBATION-THEORY SERIES FOR THE SCATTERING MATRIX

MATHEMATICAL PHYSICS

1969

SovietRxiv

## Abstract

## Full Text

UDC 530.145.1+517.948.32/35

*MATHEMATICAL PHYSICS*

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# ON EQUATIONS ARISING IN THE SUMMATION OF PERTURBATION-THEORY SERIES FOR THE SCATTERING MATRIX

*(Presented by Academician N. N. Bogolyubov on 24 II 1969)*

1. We consider a model of a scalar real field with interaction Lagrangian:

$$\mathcal{L}_{\text{int}}(x) = g : \varphi^4(x) :, \qquad x = (x^0, x^1, x^2, x^3),$$

where $\varphi(x)$ is a free field. In this model the $S$-matrix is given by the expression

$$S = T \exp\left( ig \int_{-\infty}^{\infty} : \varphi^4(x) : \, dx \right). \tag{1}$$

Expression (1) admits an expansion in normal products of the field $\varphi(x)$

$$S = F_0 + \sum_{i=1}^{N} \int \cdots \int F_N(x_1, \dots, x_N) : \varphi(x_1) \cdots \varphi(x_N) : \, dx_1 \cdots dx_N. \tag{2}$$

The functions $F_N$ are infinite series of contributions from Feynman diagrams with $N$ external lines $(N = 0, 1, 2, \dots)$ ([1,2]), and the problem arises of assigning mathematical meaning to these series.

The following scheme is proposed for solving this problem. From the formal expressions (1) and (2), relations between the functions $F_N$ are derived. These relations are considered in the Euclidean domain as a single equation in a certain Hilbert space. By the sum of the series representing $F_N$ we mean the solution of the equations obtained ([3,4]).

2. Two kinds of relations between the functions $F_N$ are possible.

I. Substituting series (2) into the relation of D. A. Kirzhnits ([5])

$$\frac{d}{dg} S = \frac{i}{g} T \int (\mathcal{L}_{\text{int}}(x) S) \, dx, \tag{3}$$

reducing the right-hand side of (3) to normal form, varying $N$ times with respect to the field $\varphi(x)$, and averaging over the vacuum, we obtain relations between

Fig. 1

Figure 1: Fig. 1

the functions

$$F_N(p_1,\dots,p_N)=\int dx_1\cdots dx_N\exp\left(i\sum_{i=1}^N p_i x_i\right)\sum_{\mathrm{perm}(x_1\cdots x_N)}F_N(x_1,\dots,x_N)$$

in the Euclidean domain:

$$\frac{d}{dg}F_N(p_1,\dots,p_N)=\sum_{s=-2}^{2}\sum_{i_1+\dots+i_{2+s}=1}\binom{4}{2+s}\prod_{l=1}^{2-s}\int\frac{dk_l}{(2\pi)^4(k_l^2+m^2)}(2\pi)^4\delta(p_{i_1}+\cdots$$
$$\cdots+p_{i_{2+s}}-k_1-\cdots-k_{2-s})F_{N-2s}(k_1,\dots,k_{2-s},p_1,\dots,\check{p}_{i_1},\dots,\check{p}_{i_{2+s}},\dots,p_N). \tag{4}$$

for

$$\sum_{i=1}^N p_i=0$$

and the initial condition $F_0(0)=1,\ F_N(0)=0$.

Equations (4) are represented graphically as follows:

**Fig. 1**

II. Acting analogously with the relation (1):

$$\frac{\delta}{\delta\varphi(x)}S=iT\left(\frac{\partial\mathcal{L}_{\mathrm{int}}(x)}{\partial\varphi(x)}S\right),$$

we obtain

$$\begin{aligned}F_N(p_1,\dots p_N)=4g\,\frac{1}{N}\Bigg(&\sum_{s=-1}^{2}\sum_{i_1\neq\dots\neq i_{2+s}=1}^{N}(1+s)^3\prod_{l=1}^{2-s}\int\frac{dk_l}{(2\pi)^4(k_l^2+m^2)}\\&\times(2\pi)^4\delta\big(p_{i_1}+\dots+p_{i_{2+s}}-k_1-\dots-k_{2-s}\big)\\&\times F_{N-2s}\big(k_1,\dots k_{2-s},p_1,\dots\check{p}_{i_1},\dots\check{p}_{i_{2+s}},\dots p_N\big)\Bigg)\end{aligned} \tag{5}$$

for

$$\sum_{i=1}^N p_i=0.$$

Equations (5) in graphical form:

Fig. 2

Figure 2: Fig. 2

**Fig. 2**

In equation (5), $F_0$ enters only when $N = 4$. Equations for the functions $F'_N = F_N/F_0$ ($F'_N$ do not contain vacuum loops) can be obtained from (5) if one sets $F_0 = 1$ (the prime is omitted below). Equations (4) and (5) were first obtained in works by one of the authors and were studied in detail in the $\varphi^3$ model ([6]).

3. Consider the Hilbert space

$$H = \sum_{N=0}^{\infty} H_N$$

of sequences $f = \{f_N\}_{N=1}^{\infty}$ of symmetric functions $f_N(p_1, \dots p_N)$ with scalar product

$$(f, g) = \sum_{N=0}^{\infty} (f_N, g_N); \qquad (f_N, g_N) = \int \bar{f}_N g_N \, dP_N, \qquad dP_N = \prod_{l=1}^{N} \frac{dp_l}{(2\pi)^4 (p_l^2 + m^2)}.$$

Introduce formal operations acting from one $H_N$ into another:

$$\begin{aligned}(a_s f)_N(p_1, \dots p_N) = \sum_{i_1 \neq \dots \neq i_{2+s}=1}^{N} \int & dP_{2-s} \, (2\pi)^4 \delta\Big(p_{i_1} + \dots + p_{i_{2+s}} \\ & - k_1 - \dots - k_{2-s}\Big) \, f_{N-2s}\Big(k_1, \dots k_{2-s}, p_1, \dots \check{p}_{i_1}, \dots \check{p}_{i_{2+s}}, \dots p_N\Big), \\ & s = 0, \ \pm 1, \ \pm 2.\end{aligned} \tag{6}$$

We pass to new functions $F_N$. In equations I,

$$F_N \to \frac{1}{\sqrt{N!}} F_N;$$

in equations II

$$F_N \to \frac{1}{\sqrt{(N-1)!}} F_N. \tag{7}$$

Let us rewrite equations I and II, using operations (6) and the substitution (7):

$$\text{I.}\quad \frac{d}{dg}F_N(g) = \Big\{\sqrt{(N+1)(N+2)(N+3)(N+4)}\,(a_{-2}F)_N(g) + \frac{1}{\sqrt{N(N-1)(N-2)(N-3)}}\,(a_2F)_N(g)\Big\} + 4\left\{\sqrt{(N+1)(N+2)}\,(a_{-1}F)_N(g) + \frac{1}{\sqrt{N(N-1)}}\,(a_1F)_N(g)\right\} + 6(a_0F)_N(g),$$

or, more abstractly:

$$\frac{d}{dg}F(g) = BF(g), \qquad F(0) = 1.$$

$$\text{II.}\quad F_N = 4g\left\{\frac{\sqrt{(N+1)N}}{N}(a_{-1}F)_N + \frac{3}{N}(a_0F)_N + \frac{3}{N\sqrt{(N-1)(N-2)}}(a_1F)_N\right\} + 4g\,\frac{1}{N\sqrt{(N-1)(N-2)(N-3)(N-4)}}(a_2F)_N + F_4^0\delta_{N4},$$

or, more abstractly:

$$F = AF + F^0, \qquad F^0 = \delta_{N4}\frac{4!}{\sqrt{3!}}\,g(2\pi)^4\delta(p_1 + \cdots + p_4).$$

Operations (6), just like the operations $A$ and $B$, have no operator meaning, but generate unbounded bilinear forms. Let us dwell on the properties of these bilinear forms.

**Lemma 1.** For infinitely differentiable functions $f_N$ and $g_N$ with compact support, the relations

$$(N-2+s)!\,(g_N, (a_{-s}f)_N) = (N-2-s)!\,((a_sg)_N, f_N), \qquad s = \pm 2, \pm 1, 0$$

hold.

**Proof** follows from the definition of the operations $a_s$.

**Lemma 2.** If the components of the columns $f$ and $g$ satisfy the condition of Lemma 1, then the bilinear form $(g, Bf)$ is symmetric.

**Remark.** The bilinear form $(g, Af)$ is nonsymmetric because of the absence of the operation $a_{-2}$.

4. Let us smooth our equations and operations (6), introducing into them a form factor by means of the substitution

$$\delta(p_1+\dots+p_4) \to \varphi(p_1,\dots,p_4) \in H_4, \qquad \operatorname{Im}\varphi = 0,$$

$$\varphi(p_1,p_2,p_3,p_4) = \varphi(-p_1,-p_2,-p_3,-p_4).$$

Thus we have $a_s \to a_s(\varphi)$, $B \to B(\varphi)$. Such operators are already well defined. The following convinces us of this.

**Lemma 3.** The estimates

$$\|(a_s(\varphi)f)_N\| \le \frac{N!}{(N-2-s)!}\,\|\varphi\|\,\|f_{N-2s}\|, \qquad s=\pm2,\pm1,0$$

hold.

**Proof** follows from the definition of the operators $a_s(\varphi)$ and Schwarz' s inequality.

**Corollary.** The operator $B(\varphi)$ is unbounded. It is an operator-valued Jacobi matrix (7) and is defined on the everywhere dense set in $H$ of finite columns.

**Lemma 4.** The following conjugation properties hold:

$$(N-2+s)!\,(g_N,(a_{-s}(\varphi)f)_N) = (N-2-s)!\,((a_s(\varphi)g)_N,f_N),$$

$$s=\pm2,\pm1,0.$$

**Lemma 5.** *The operator $B(\varphi)$ is symmetric on the set of finite columns.*

**Corollary.** *The operator $B(\varphi)$ is real. Therefore, by Lemma 5, it admits a self-adjoint extension.*

Denote the smoothed equation I by $I_\varphi$. Then the following holds.

**Theorem 1.** *A solution of equation $I_\varphi$ exists in $H$ and is given by the equality*

$$F(g) = \exp(gB'(\varphi))F(0), \quad \operatorname{Re} g = 0, \quad F(0)=1,$$

*where $B'(\varphi)$ is some self-adjoint extension of the operator $B(\varphi)$.*

**Proof** follows from the theorem on the representation of functions of an unbounded self-adjoint operator.

5. Equations (4) possess one remarkable property: the operator $B$ that defines them is a symmetric Jacobi operator-valued matrix. Let us recall in this connection that the Lippmann–Schwinger and Bethe–Salpeter equations in the Euclidean region are likewise defined by a symmetric operator.

By contrast, the operator $A$ defining equations (5) (they can also be obtained from Schwinger equations [8–10]) is not symmetric. Equations (4), moreover, are more meaningful physically: they completely restore the entire structure of the $S$-matrix. It seems to us that first one must solve equations (4) and normalize the solutions to the vacuum multipliers.

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Received
14 II 1968

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*Note: Figure translations are in progress. See original paper for figures.*