Determination of the Law of Turbulent Friction in the Core of a Flow on the Basis of the Principle of Maximum Stability

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Determination of the Law of Turbulent Friction in the Core of a Flow on the Basis of the Principle of Maximum Stability

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In paper (1) a principle of maximum stability of averaged turbulent flows was proposed. On the basis of this principle it is proposed to select the mean-velocity profile of a turbulent flow from a broad class of profiles by requiring maximum stability with respect to arbitrary small disturbances. Such an approach has already yielded definite results. In particular, in studying flow in a plane-parallel channel it proved possible to calculate theoretically the Kármán constant of wall turbulence (1).

It is known (2) that in the theory of hydrodynamic stability the decisive influence on the development of a small disturbance is exerted by the position of the so-called critical point y_c, where the phase velocity of propagation of the disturbance coincides with the local velocity of the flow.

Investigations carried out by the authors have shown that, at large Reynolds numbers Re, small disturbances can be divided into two classes. The first class consists of disturbances with wave numbers $\alpha \gg 1$. They are practically different from zero only in a small neighborhood of the point y_c and are responsible for the stability only of that portion of the mean-velocity profile where they are localized. If the profile is arbitrarily deformed far from this portion, this will not affect the behavior of short-wave disturbances. Conversely, an instability associated with a local roughness of the mean-velocity profile will be caused by the rapid growth of a short-wave disturbance having a critical point on this portion of the profile. The second class consists of long-wave disturbances with $\alpha \lesssim 1$. For them localization near y_c is not characteristic, and they are responsible for the stability of the mean-velocity profile as a whole, integrally, ignoring small local roughnesses.

In paper (1), short-wave disturbances localized in the wall region and long-wave disturbances responsible for global stability were considered. A small deformation of the mean-velocity profile in the core of the flow could not affect the results obtained, and thus the latter concern only wall turbulence, but do not control the form of the averaged-velocity profile in the core.

But if the principle of maximum stability is correct, then the requirement of the

Fig. 1

Figure 1: Fig. 1

existence of short-wave disturbances localized in the core region of the flow and responsible for its stability must necessarily follow. And indeed, extensive numerical experiments carried out by the authors indicate that, for mean-velocity profiles having no inflection points, short-wave disturbances are localized on two portions of the profile—in the wall region and in the region of the maximum of the mean velocity. Moreover, among these short-wave disturbances there exists the most "dangerous" disturbance of a certain finite wavelength, which decays more slowly than the others. Thus, the possibility arises

consider the problem of the maximally stable form of the corresponding local segment of the profile, for example, the near-axis one.

Taking into account the above-mentioned property of locality, it is expedient to choose as the measure of stability the functional $\max_\alpha Y \equiv \Pi$, proposed in ([1]). (Here Y is the imaginary part of the complex propagation velocity of disturbances $C = X + iY$. The case $Y < 0$ corresponds to the damping of a small disturbance. X is the phase velocity of the disturbance.) In this case, from the entire spectrum of small disturbances the spectral number corresponding to the most slowly damped mode is selected.

For selecting the class of profiles to be tested for stability, the Prandtl–Taylor law for turbulent friction is used in the generalized form

$$\left(-\frac{l}{v_*}\frac{du}{dy}\right)^k = \frac{y}{h}. \tag{1}$$

Here $u(y)$ is the profile of the mean velocity, v_* is the dynamic velocity, h is the half-width of the channel, l is the mixing length, and y is the distance from the middle of the channel. The quantity l in the near-axis zone is assumed constant. Considerations on this point are set forth in ([3]). Consequently, the velocity profile will have the form

$$u = u_{\max} - v_*(h/l)(y/h)^n/n, \tag{2}$$

where $n = 1 + 1/k$.

Fig. 1

Relation (2) may be interpreted as the principal part of the expansion of an arbitrary profile near the axis into a generalized power series. The quantity $\bar{u}_{\max}$ is connected with the choice of the coordinate system and does not affect the stability. Therefore, without loss of generality, one may write, passing to dimensionless variables,

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

$$u = 1 - y^n/n. \tag{3}$$

The characteristic length chosen is h, and the characteristic velocity is the quantity $u_0 = v_* h/l$; n is a variable parameter to be determined from the requirement of maximum stability. (Since for $n < 2$, u'' becomes infinite at $y = 0$, in order to avoid inconveniences connected with machine computation, $y + 10^{-9}$ was used instead of y in formula (3).)

If, as before ([1]), it is assumed that the turbulent-stress tensor is not disturbed in the first approximation, then the complex amplitude $\varphi(y)$ of the disturbed stream function will satisfy the Orr–Sommerfeld equation

$$\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha \operatorname{Re}\left[(u - C)(\varphi'' - \alpha^2\varphi) - u''\varphi\right], \tag{4}$$

where $\mathrm{Re} = v_* h^2/\nu l$.

The latter assumption is apparently all the more acceptable for the case of large α, when the scales of the disturbance are smaller than the dimensions of the pulsations responsible for the turbulent transport of momentum.

Figure 1 presents the dependence $Y(\alpha)$ for the most dangerous mode of near-axis disturbances. (Dash-dotted lines show the asymptotic dependences for very large and very small α.) It is interesting to note that $Y(\alpha)$ has two local maxima. The smaller of them is located at $\alpha \sim 1$, and has a critical layer far from the axis, as is seen from Fig. 2, where the dependence of Y on the position of the critical point y_c is plotted. By virtue of the stated properties of locality, it cannot correspond

for the detailed structure of the near-axis region. The larger maximum corresponds to $a \sim 20$ and $y_c = 0.06$; consequently, it is precisely this maximum that controls the stability of the near-axis region. The data presented in Figs. 1 and 2 correspond to $\mathrm{Re} = 10^4$ and $n = 1.5$.

Fig. 2

Fig. 3

In Fig. 3 the dependence of the value of the last maximum

Fig. 4

Figure 4: Fig. 4

$$\Pi = \max_a Y$$

on n is shown for different Reynolds numbers. Curves 1, 2, 3, 4 correspond respectively to $\mathrm{Re} = 10^3,\ 10^4,\ 10^5,\ 10^6$. It is easy to see that profiles with $n < 1.12$ are unstable at these Reynolds numbers ($\Pi > 0$), while the maximum stability is observed at $n \simeq 1.5$. This corresponds to the well-known experimental Darcy law for the mean velocity in the turbulent core and to the quadratic dependence of turbulent friction on the gradient of the mean velocity (the Prandtl–Kármán law) ([4]). Thus, the principle of maximum stability in this case as well makes it possible to obtain theoretically the known experimental dependences.

Fig. 4

It may be noted that the value n corresponding to the maximally stable profile depends on Re, although only very weakly. Figure 4 shows the dependence $n(\mathrm{Re})$ over a broad range of variation of the Reynolds number. This dependence, along with numerical calculations, can be obtained from similarity considerations.

Indeed, let some λ be chosen as the characteristic length, and let $u_0(\lambda/h)^n$ be chosen as the characteristic velocity. Then the dimensionless velocity profile retains the form (4), but the role of the Reynolds number will be played by the combination $u_0\lambda(\lambda/h)^n/\nu = \mathrm{Re}_1$. By virtue of locality properties, the behavior of short-wave disturbances with a critical point at the axis will not depend on the new dimensionless channel width, and if λ is chosen so that $\mathrm{Re}_1 = 1$, i.e.

$$\lambda/h = \mathrm{Re}^{-1/(n+1)}, \tag{5}$$

then the quantity $F = \Pi/(\lambda/h)^n$, corresponding to the functional Π in the new scales, will be a function only of the number n.

The function $F(n)$ can be calculated from any of the curves in Fig. 3, and then n, which ensures maximum stability, is found as the root of the equation

$$(n+1)^2 F'(n)/F(n) = \ln \mathrm{Re}\,. \tag{6}$$

Calculations by formula (6) are in good agreement with the dependence presented in Fig. 4.

It should be noted that relation (5) estimates the size of the near-axis portion of the profile, which has a substantial influence on the behavior of short-wave disturbances localized at the axis. Accordingly, the α corresponding to the most dangerous disturbance is proportional to $\mathrm{Re}^{1/(n+1)}$. Thus, as the Reynolds number increases, an increasingly narrow near-axis portion of the velocity profile is determined in this way.

It is known that in the immediate vicinity of the channel axis the relation for turbulent friction deviates from the quadratic law. Attempts have been made to describe it, for example, by means of a representation of the form [4]

$$\tau = \rho l^2 (du/dy)\sqrt{(du/dy)^2 + l_1^2 (d^2u/dy^2)^2}. \tag{7}$$

It is also well known that power-law representations for the mean profile of turbulent velocity, both in the wall region and in the core region, are not universal with respect to the Reynolds number.

These facts are fully consistent with the result obtained here, namely that at very large Re, n deviates somewhat from one and a half. The deviation from the quadratic law of friction at small Re is evidently associated with the degeneration of turbulence. Nevertheless, over practically the entire experimental range of Reynolds numbers for turbulent flows, the exponent n differs only very weakly from 3/2.

Thus, the principle of maximum stability makes it possible to predict theoretically the law of turbulent friction in the core of the flow (or the distribution of the averaged turbulent velocity at the channel axis) in complete agreement with known experimental data.

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