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Abstract

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THE AVERAGING PRINCIPLE AND BIFURCATION OF ALMOST PERIODIC SOLUTIONS

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As is known, oscillatory systems in many cases possess one or several equilibrium states. It is of interest to determine the conditions under which, when the parameters of the system are varied, periodic or almost periodic oscillations are born from a given equilibrium state. For the case of periodic oscillations this problem was investigated in the works of M. A. Krasnosel' skii, Yu. I. Neimark, and others by topological methods. The analogous problem for almost periodic oscillations is connected with overcoming substantial difficulties. The point is that the integral equations whose solutions determine the almost periodic oscillations of the system are not equations with completely continuous operators and, moreover, the linearized operator corresponding to the equilibrium state is not normally solvable. Thus, for the study of the birth of almost periodic oscillations the classical theory of branching and the general theory of bifurcation points turn out to be inapplicable.

In the present note the problem of the birth of almost periodic oscillations is studied by methods of the theory of simple solutions (see, for example, (^{1})) and by methods of the theory of cones (^{2}). As an example, the question of the birth of almost periodic oscillations for a pendulum with a vibrating suspension point is considered.

1. Consider in the real n-dimensional space E_n the equation

$$dx/dt = \varepsilon A_1(t)x + \ldots + \varepsilon^k A_k(t)x + \varepsilon^{k+1} A(t,\varepsilon)x + \varepsilon F(t,x,\varepsilon) + \varepsilon\omega(t,x,\varepsilon). \quad (1)$$

Here $A_1(t), \ldots, A_k(t)$ are square matrices of order n, the elements of which are quasiperiodic polynomials; $A(t,\varepsilon)$ is a square matrix almost periodic in t, depending continuously on ε uniformly with respect to $t \in (-\infty, \infty)$; $F(t,x,\varepsilon)$ is a form of order m in the spatial variable x, the coefficients of which are quasiperiodic polynomials depending continuously on ε uniformly with respect to $t \in (-\infty, \infty)$; $\omega(t, x, \varepsilon)$ $(\omega(t, 0, \varepsilon) \equiv 0)$ is an operator containing, in x, terms of order of smallness higher than m:

$$\|\omega(t, x_1, \varepsilon) - \omega(t, x_2, \varepsilon)\| \leq q(r)\|x_1 - x_2\| \quad (\|x_1\|, \|x_2\| \leq r),$$

where

$$\lim_{r \to 0} r^{-m+1} q(r) = 0,$$

almost periodically depending on t and continuously depending on ε uniformly with respect to $t \in (-\infty, \infty)$. The corresponding averaged equation of the first approximation has the form

$$dx/d\tau = B_1 x + \overline{F}(x) + \overline{\omega}(x) \qquad (\tau = \varepsilon t), \tag{2}$$

where

$$B_1 = \lim_{T \to \infty} \frac{1}{T} \int_0^T A_1(t)\, dt,$$

$$\overline{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t, x, 0)\, dt, \qquad \overline{\omega}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \omega(t, x, 0)\, dt.$$

Suppose that 0 is a simple eigenvalue of the matrix B_1, and that all its remaining eigenvalues lie in the left half-plane. Denote by e_0 and g_0 such eigenvectors of the matrices B_1 and B_1^*, respectively, corresponding to this eigenvalue, that $(e_0, g_0) = 1$. We shall assume that the number

$$c = (\overline{F}(e_0), g_0) \tag{3}$$

is different from zero.

Let us now consider the auxiliary linear system

$$dx/dt = \varepsilon A_1(t)x + \ldots + \varepsilon^k A_k(t)x + \varepsilon^{k+1} A(t, \varepsilon)x. \tag{4}$$

By means of the known transformations of N. N. Bogolyubov–I. Z. Shtokalo, this system can be brought to a system autonomous up to terms of order k:

$$dx/dt = \varepsilon B_1 x + \ldots + \varepsilon^k B_{kx} + \varepsilon^{k+1} B(t, \varepsilon)x; \tag{5}$$

here the matrices $B_2, \ldots, B_k$ are determined by recurrent formulas (see (^{3})). Denote by $\mu(\varepsilon)$ the eigenvalue of the matrix

$$B(\varepsilon) = B_1 + \varepsilon B_2 + \ldots + \varepsilon^{k-1} B_k,$$

which tends to zero as $\varepsilon = 0$; we shall assume that

$$\mu(\varepsilon) = a\varepsilon^{k_0} + o(\varepsilon^{k_0}), \tag{6}$$

where $k_0 \leq k - 1$ and $a \neq 0$. Recall (^{3}) that for $a < 0$ the zero solution of equation (4) is exponentially stable, while for $a > 0$ it is unstable.

Theorem 1. *There exist positive numbers* ε_0 *and* r_0 *such that the following assertions are valid:*

1°. Equation (2) has no almost periodic solutions in the ball $\|x\| \leq r_0$.

2°. If m *is even, then for* $\varepsilon \in (0, \varepsilon_0)$ *equation (1) has in the ball* $\|x\| \leq r_0$ *a unique nonzero almost periodic solution; this solution is exponentially stable for* $a > 0$ *and unstable for* $a < 0$.

3°. If m *is odd and* $ac < 0$*, then for* $\varepsilon \in (0, \varepsilon_0)$ *equation (1) has in the ball* $\|x\| \leq r_0$ *exactly two nonzero almost periodic solutions; these solutions are exponentially stable for* $a > 0$ *and unstable for* $a < 0$.

4°. If m *is odd and* $ac > 0$*, then for* $\varepsilon \in (0, \varepsilon_0)$ *equation (1) has no nonzero almost periodic solutions in the ball* $\|x\| \leq r_0$.

2. Theorem 1 is modified in a natural way in the case where all functions in equation (1) are ω-periodic in t. In this case a bifurcation occurs in the class of ω-periodic solutions. It should be noted that the problem of periodic solutions is substantially simpler than the problem of almost periodic solutions.

3. As an example, consider the equation

$$\ddot{\theta} + 2a\varepsilon\dot{\theta} + [\varepsilon^2 k^2 - \varepsilon p(t)] \sin\theta = 0 \tag{7}$$

of oscillations of a pendulum with a vibrating point of suspension. Here $a > 0$, $p(t)$ is a quasiperiodic polynomial with zero mean, and ε is a positive parameter. For small ε the lower equilibrium state ($\theta = 0$) is always

stable. The upper equilibrium state ($\theta = \pi$), as shown by N. N. Bogolyubov (^{4,5}) and P. L. Kapitsa (^{6}), may, for small ε, be either stable or unstable, depending on the function $p(t)$.

Following N. N. Bogolyubov, by means of the substitution

$$\theta = x_1 + \pi - \varepsilon p(t) \sin x_1, \qquad \dot{\theta} = \varepsilon x_2 - \varepsilon p'(t) \sin x_1 \tag{8}$$

one can pass to a standard equation of the form (1), in which x is a two-dimensional vector, $m = 3$, and

$$B_1 = \begin{pmatrix} 0 & 1 \\ -M({p'}^2) + k^2 & -2\alpha \end{pmatrix}. \tag{9}$$

Here $M({p'}^2)$ denotes the mean value of the function ${p'}^2(t)$. If $M({p'}^2) > k^2$, then the upper equilibrium state is stable; if $M({p'}^2) < k^2$, then the upper equilibrium state is unstable. Moreover, in a neighborhood of the upper equilibrium state, for sufficiently small ε, there are no almost periodic oscillations.

We are interested in the case when

$$M({p'}^2) = k^2. \tag{10}$$

In this case the investigation of the stability of the upper equilibrium state can, generally speaking, be carried out after the successive computation of a finite number of matrices B_2, B_3, and so on (see ([3])). We shall assume that such an investigation has been carried out. Then the following assertions follow from Theorem 1.

Theorem 2. Suppose that condition (10) is satisfied. Then:

1°. If the upper equilibrium state of the pendulum is unstable, then in its neighborhood there are no almost periodic oscillations.

2°. If the upper equilibrium state of the pendulum is stable, then in each of its neighborhoods, for small ε, there are exactly two almost periodic oscillations, and these are unstable.

A more detailed analysis shows that, in the case when condition (10) is satisfied and when the upper equilibrium state is exponentially stable, there exists a neighborhood U_0 of the upper equilibrium state such that, for any other fixed neighborhood V, for every sufficiently small ε there are initial conditions in V to which there correspond solutions $\theta = \theta(t)$ $(t \geq 0)$ of equation (7) that do not lie wholly in U_0. This fact means that, under condition (10), the upper equilibrium state is "practically unstable" (see ([7])).

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Note: Figure translations are in progress. See original paper for figures.

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