Soviet-era science, translated into English

DERIVATION OF THREE-DIMENSIONAL SIMILARITY SYMMETRY GROUPS

CRYSTALLOGRAPHY

1968

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196801.99965

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.45

CRYSTALLOGRAPHY

A. M. ZAMORZAEV

DERIVATION OF THREE-DIMENSIONAL SIMILARITY SYMMETRY GROUPS

(Presented by Academician A. V. Shubnikov, May 26, 1967)

1. In 1960 A. V. Shubnikov proposed the idea of similarity symmetry ([1]). A refinement of the concept of similarity symmetry groups, the transfer to them of the idea of antisymmetry ([2]), a description of two-dimensional crystallographic groups of similarity symmetry and antisymmetry, and their generalization with the concept of antisymmetry of various kinds ([3]) are given in ([4]). Three-dimensional groups of similarity symmetry and antisymmetry have been studied in part ([5,6]).

A group G of similarity transformations in n-dimensional Euclidean space will be called an n-dimensional similarity group (s.g.) if: a) G contains at least one similarity transformation P with coefficient $k \neq 1$; b) at least one point of the space is isolated in the infinite class of its images under all transformations from G ([5]). The point O, fixed under the transformation P, will be called a special point of the group G; it is fixed under all transformations of the group G ([4,5]).

The derivation of two-dimensional groups of similarity symmetry and antisymmetry in ([4]) is reduced to the consideration of symmetry and antisymmetry groups of rods ([7]). The generalization of these and other groups with the concept of antisymmetry of various kinds was carried out in parallel ([4,8]). Two-dimensional groups of similarity symmetry and antisymmetry have been applied to the description and derivation of a special class of three-dimensional s.g.' s—the so-called conical ones (three-dimensional with a special plane) ([4,6]). Crystallographic conical s.g.' s have been completely generalized with the concept of antisymmetry of various kinds ([6]). The study of noncrystallographic conical and two-dimensional s.g.' s is also reduced to consideration of the better-known noncrystallographic rod groups (symmetry of chain molecules ([9,10])).

The central content of the present work is the derivation of all three-dimensional s.g.' s. By the method of generalized projections in the spirit of ([11]), they can be interpreted as point groups of symmetry and antisymmetry ([2,12]) and of color symmetry ([13]). Many general properties of s.g.' s are valid for any dimension n.

SovietRxiv

2. Let G be an n-dimensional s.g. with special point O. Every transformation g of the group G is uniquely representable as the product of a homothety h, with center O, and a transformation of classical symmetry s, keeping O fixed (a "rotation" about O), which we shall call the components of g. It is easy to see that the collections H and S, respectively, of the homotheties and "rotations" occurring in the transformations of the group G, are also groups.

Construct mappings φ and ψ of the group G onto S and onto H according to the rule $\varphi(g) = s$, $\psi(g) = h$, where $g = hs$ $(g \in G)$. Obviously, φ and ψ are homomorphisms, whose kernels are the subgroup H_0 of all homotheties in G $(H_0 = G \cap H)$ and the subgroup S_0 of all "rotations" in G $(S_0 = G \cap S)$. By the homomorphism theorem (^{14}), H_0 and S_0 are normal divisors of the group G, and moreover $G/H_0 \cong S$ and $G/S_0 \cong H$.

Theorem 1. *Every s.g. contains a transformation with the least similarity coefficient exceeding unity.*

The given group G, by property a), contains a transformation with coefficient $k > 1$, and by b) there exists a point A (different from O) whose ε-neighborhood

which contains no other points of the class of its G-images. Every transformation g from G with coefficient $k' > 1$ carries this ε-neighborhood into an ε'-neighborhood of the point $A' = g(A)$, containing no other points of this class, with $\varepsilon' > \varepsilon$. Therefore, in the layer between the spheres of radii OA and $k \cdot OA$ with center O, the point A' can assume only a finite number of positions; hence the number of such values k', $1 \leq k' < k$, is finite, and one may choose the smallest of them.

From the theorem just proved it follows that: 1) the group H of homotheties occurring among the transformations of the group G is cyclic (generated by a homothety with the smallest coefficient $k' > 1$); 2) if the subgroup H_0 of homotheties of the group G is nontrivial, then it is cyclic and its index in the group H is finite.

Let us now note that, for $n \geq 3$, the definition given in Section 1 is satisfied not only by discrete groups, but also by groups of semicontinua (for example, the three-dimensional finite group $\{K\}(\infty : m)$ in the notation of (^{6})). Therefore we shall replace requirement b) by a stronger one: b′) there exist n points not lying in one $(n-2)$-dimensional plane, each of which is isolated in the class of its G-images*. In what follows we consider only groups defined by requirements a) and b′).

Theorem 2. *If a subgroup G_0 of the group G is itself an f.s.g., then the index of G_0 in G is finite.*

We give the proof for three-dimensional groups.

By property a), G_0 contains a transformation with coefficient $k > 1$. By b′), for G there exist noncollinear points A, B, C, isolated in the classes of their

G-images; two of them, for example A and B, are noncollinear with O. Every adjacent class gG_0 contains such a transformation g' that

$$OA \leq OA' < k \cdot OA, \qquad OB \leq OB' < k \cdot OB, \quad \text{where } A' = g'(A), \quad B' = g'(B). \tag{*}$$

Reasoning with spherical layers, as in Theorem 1, we find that for $g' \in G$ there are only a finite number p of points A' and a finite number q of points B' satisfying condition $(*)$. Consequently, the triangle $OA'B'$ (the g'-image of the triangle OAB) assumes no more than pq positions; but a similarity transformation in space is determined two-valuedly by specifying a triangle and its image (^{15}), and therefore there are no more than $2pq$ distinct g' satisfying $(*)$. Hence the number of adjacent classes gG_0 is finite.

For any $n > 3$ the basic idea of the argument is exactly the same.

From this theorem it follows that: 1) the subgroup S_0 of "rotations" of the group G is finite (if $g' \in S_0$, then it satisfies condition $(*)$); 2) if G has a nontrivial subgroup of homotheties H_0, then G/H_0 and the group S of "rotations"occurring among the transformations of G are finite.

3. Let now G be an f.s.g. with a nontrivial subgroup of homotheties H_0. Then, by Section 2, G/H_0 is finite; $H_0 = \{K\}$, where K is a homothety from G with the smallest coefficient $k > 1$; every adjacent class gH_0 contains the unique transformation g' satisfying condition $(*)$ for any points A and B distinct from O, and its coefficient is equal to one of the numbers $1, k^{1/p}, \dots, k^{p-1/p}$ (p is the index of the subgroup H_0 in H). Choose a sphere of radius R with center O, and on it a point A; the class of its H_0-images consists of the intersections of the ray OA with all spheres of radii Rk^l (where l is any integer) having center O; the points A' satisfying $(*)$ will lie on the concentric spheres of radii $R, Rk^{1/p}, \dots, Rk^{p-1/p}$. Project the points A' and their H_0-images (and these exhaust all G-images of the point A) from the center O onto the chosen sphere of radius R, and color their projections in p different colors corresponding to the p "spherical levels" of the points A': to the point $\bar{A}$, which is the projection of the point A' (and of its H_0-images),

---

* By virtue of a), these classes are infinite for at least $n-1$ of these points; therefore for $n \geq 2$, b$'$) implies b).

we assign the j-th color if $OA' = Rk^{j-1/p}$ ($j = 1, 2, \dots, p$). Any transformation from the coset gH_0 maps the class of H_0-images of the point A into the class of H_0-images of the point $A' = g'(A)$, where $g' \in gH_0$; therefore, in colored projections onto the sphere the class gH_0 is represented by a transformation of p-colored symmetry in the sense of Belov (^{11}) (for $p = 2$, "black-and-white,"for $p = 1$, classical), carrying A into $\bar{A}$ and geometrically coinciding with the "rotations" entering into the transformations of the class gH_0. The totality $\widetilde{S}$ of such transformations is a point group of p-colored symmetry, isomorphic to G/H_0 and obtained from the group S of "rotations" entering into the transformations G; $\widetilde{S}$ is a natural model of G/H_0 and thereby interprets G (cf. the interpretation of the three-dimensional Fedorov groups in the form of colored two-dimensional methods of generalized projections of Belov-Tarkhova (11)).

Thus, all s.s.g. containing nontrivial homotheties can be described by means of point groups of p-colored symmetry, well studied for $n = 3$ (2,12,13). Therefore the problem of deriving three-dimensional s.s.g. with homotheties is in fact completely solved. All crystallographic s.s.g. (for which S is one of the 32 crystallographic point groups (2,12)) are easily written out by means of the corresponding p-colored point groups, assigning to p the values $1, 2, 3, 4, 6$. For $p = 1$ there are 32 of them, for $p = 2$ 58 (12), for $p = 3, 4, 6$ there are $7{+}4{+}7 = 18$ (13), and in all 108. Of these, 94 groups are known as 109 conical groups (4,6) (in Table 2 of work (6), 15 conical groups repeat those derived earlier, if they are regarded as three-dimensional without taking into account a distinguished direction). The remaining 14 s.s.g., associated with the cubic system, do not fall into the conical series. Let us write them in the notation of (4,6), using for point groups of symmetry, antisymmetry, and color symmetry the notation of (2,13).

The groups $3/2$ and $\bar{3}^{(3)}/2$ are interpreted by the s.s.g. $\{K\}(3/2)$ and $\{K\}({K'}^{1/3}\bar{3}/2)$; the groups $\bar{6}/2$, $\bar{6}/2$, $\bar{6}^{(3)}/2$, and $\bar{6}^{(6)}/2$—by the s.s.g. $\{K\}(\bar{6}/2)$, $\{K\}({K'}^{1/2}\bar{6}/2)$, $\{K\}({K'}^{1/3}\bar{6}/2)$, and $\{K\}({K'}^{1/6}\bar{6}/2)$; the groups $3/\bar{4}$, $3/\bar{4}$, $3/4$, and $3/\bar{4}$–$\{K\}(3/\bar{4})$, $\{K\}(3/{K'}^{1/2}\bar{4})$, $\{K\}(3/4)$, and $\{K\}(3/{K'}^{1/2}4)$; the groups $\bar{6}/4$, $\bar{6}/\bar{4}$, $\bar{6}/4$, and $\bar{6}/\bar{4}$–$\{K\}(\bar{6}/4)$, $\{K\}({K'}^{1/2}\bar{6}/4)$, $\{K\}(\bar{6}/{K'}^{1/2}4)$, and $\{K\}({K'}^{1/2}\bar{6}/{K'}^{1/2}4)$.

Among noncrystallographic three-dimensional s.s.g. with homotheties, three are not conical: $\{K\}(3/5)$, $\{K\}(3/10)$, and $\{K\}(3/{K'}^{1/2}\overline{10})$ (they are interpreted by the groups $3/5$, $3/\overline{10}$, and $3/\overline{10}$ (2); for $p > 2$ colored groups from $3/5$ and $3/\overline{10}$ are not derived). All the remaining three-dimensional groups with homotheties are conical, since the remaining three-dimensional point groups have a plane invariant under all their transformations (distinguished).

4. We shall now show that all three-dimensional s.s.g., with the exception of the 17 listed above, are conical. It remains to consider only groups that do not contain nontrivial homotheties (and, consequently, also homothetic reflections (4–6), and screw motions or reflections (4–6) with rotation angles, and mirror rotations entering into them, rational with respect to π).

Theorem 3. *A three-dimensional s.s.g. without nontrivial homotheties is conical.*

Let G be a three-dimensional s.s.g. without homotheties, and P a transforma-

SovietRxiv

tion contained in it with coefficient $k \neq 1$. In this case P can only be a screw motion (or screw reflection) ($^{4-6}$) with an angle of rotation (or mirror rotation) φ irrational with respect to π; it leaves invariant exactly one line d (axis). We shall show that d is a distinguished line of the group G, i.e., invariant under its transformations.

Suppose that G contains a transformation g carrying d into another line d_1. The transformation $P_1 = gPg^{-1}$ is a screw motion (reflection) with coefficient k and angle φ, but with axis d_1, and the transformations $P_2 = PP_1P^{-1}, \dots, P_n = P^{n-1}P_1P^{1-n}, \dots$ are screw motions (reflections)

with coefficient k, angle φ, and axes $d_2 = P(d_1), \dots, d_n = P^{n-1}(d_1), \dots$. Since φ/π is irrational, all the lines d_i $(i = 1, 2, \dots)$ are distinct; therefore the transformations P_i are also distinct, and hence so are all $s_i = P^{-1}P_i$, belonging to the subgroup S_0 of "rotations" of the group G. But this contradicts the finiteness of the group S_0 (Corollary 1 to Theorem 2).

Consequently, d is a special line. But then the plane perpendicular to it and passing through the point O is also special, i.e., the group G is finite.

All finite s.s.g., as was already noted in § 1, can be described by means of rod groups of symmetry and antisymmetry, whose survey presents no fundamental difficulty. The problem of deriving three-dimensional s.s.g. may be regarded as completely solved.

5. From §§ 2, 3 it follows that the problem of deriving n-dimensional s.s.g. with homotheties (in particular, crystallographic ones) comes down directly to the study of n-dimensional point groups of symmetry and the derivation from them of color groups. The case $n = 4$ is facilitated by the fact that the point crystallographic groups have already been found (^{16}).

In (^{5}) it was noted that the derivation of n-dimensional s.s.g. makes it possible at once to describe $(n+1)$-dimensional linear groups of symmetry with an invariant directed line (symmetry groups of $(n+1)$-dimensional directed rods). Thus, in the present work the problem of deriving the symmetry groups of directed 4-dimensional rods has also been solved; there are 108 crystallographic groups among them. For the description of all 4-dimensional rod groups, however, an extension of three-dimensional s.s.g. to the so-called groups of conformal symmetry (§ 5 (^{4})) is required.

Kishinev State University

Received
15 V 1967

REFERENCES

1. A. V. Shubnikov, *Kristallografiya*, **5**, no. 4, 489 (1960).

SovietRxiv

2. A. V. Shubnikov, *Symmetry and Antisymmetry of Finite Figures*, Moscow, 1951.
3. A. M. Zamorzaev, E. I. Sokolov, *Kristallografiya*, **2**, no. 1, 9 (1957).
4. E. I. Galyarskii, A. M. Zamorzaev, *Kristallografiya*, **8**, no. 5, 691 (1963).
5. A. M. Zamorzaev, *DAN*, **167**, no. 2, 334 (1966).
6. E. I. Galyarskii, *Kristallografiya*, **12**, no. 2, 202 (1967).
7. N. N. Neronova, N. V. Belov, *Kristallografiya*, **6**, no. 1, 3 (1961).
8. E. I. Galyarskii, A. M. Zamorzaev, *Kristallografiya*, **10**, no. 2, 147 (1965).
9. B. K. Vainshtein, *Diffraction of X-rays on Chain Molecules*, Moscow, 1963.
10. B. K. Vainshtein, *Kristallografiya*, **4**, no. 6, 842 (1959).
11. N. V. Belov, T. N. Tarkhova, *Kristallografiya*, **1**, no. 1, 4 (1956).
12. V. A. Koptsik, *Shubnikov Groups (Handbook)*, Moscow, 1966.
13. V. L. Indenbom, N. V. Belov, N. N. Neronova, *Kristallografiya*, **5**, no. 4, 497 (1960).
14. A. G. Kurosh, *Group Theory*, Moscow, 1953.
15. P. S. Modenov, A. S. Parkhomenko, *Geometric Transformations*, Moscow, 1961.
16. A. C. Hurley, *Proc. Cambr. Phil. Soc.*, **47**, 650 (1951).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.