ON THE BEST APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC \(C[a,b]\) BY GENERALIZED POLYNOMIALS

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Abstract

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ON THE BEST APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC $C[a, b]$ BY GENERALIZED POLYNOMIALS

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In the present paper we indicate necessary and sufficient conditions that must be imposed on a system of functions $\{\varphi_i(x)\}_1^{n+1}$ so that every polynomial

$$P(c, x) = \sum_{i=1}^{n+1} c_i \varphi_i(x)$$

least deviating in the metric $C[a, b]$ from a function $f(x) \in C[a, b]$ possess no fewer than $k+1$ $(0 \le k \le n+1)$ points of alternance.

Definition. A system of continuous and linearly independent functions $\{\varphi_i(x)\}_1^{n+1}$ on $[a, b]$ will be called a **system of type** k $(0 \le k \le n)$ if: 1) for any points $a \le x_1 < x_2 < \dots < x_{n+1} \le b$ such that

$$\begin{vmatrix} \varphi_1(x_1) & \dots & \varphi_{n+1}(x_1) \\ \dots & \dots & \dots \\ \varphi_1(x_{n+1}) & \dots & \varphi_{n+1}(x_{n+1}) \end{vmatrix} = 0,$$

there exist $i+1$ rows $(k \le i \le n)$ for which all minors of order $i+1$ are equal to zero (at least for one i); 2) in at least one such determinant of order $i+1$ all complementary minors D_j^i $(j = 1, \dots, i+1)$ of the elements of at least one column are different from zero, and in the sequence $D_1^i, \dots, D_{i+1}^i$ there are no more than $i-k$ sign changes; moreover, the set of points $\{x_i\}_1^{n+1}$ for which in this sequence (for some i) there will be exactly $i-k$ sign changes.

Examples of systems of type k:

1. Define on the segment $[-1, 2]$ continuous functions in the following way:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \dots, \quad \varphi_{k-1}(x) = x^{k-2}, \quad \varphi_k(x) = x^k,$$

$$\varphi_{k+1}(x) = x^{k+1} + a_0 x^k + \dots + a_k,$$

$$\varphi_i(x) = \begin{cases} 0 & \text{for } x \in [-1,1], \\ x^i - 1 & \text{for } x \in (1,2], \end{cases} \qquad (i = k+2, \dots, n)$$

$$\varphi_{n+1}(x) = \begin{cases} 0 & \text{for } x \in [-1,1], \\ x^{k-1} - 1 & \text{for } x \in (1,2]. \end{cases}$$

It is possible to choose the parameters $a_0, a_1, \dots, a_k$ so that no k functions from $\{\varphi_i(x)\}_1^{n+1}$ form a Chebyshev system (T-system) on $[-1,2]$, but the whole system will be of type k.

2. The system of functions $1, x^2, x^4, \dots, x^{2n}$ will be a system of type 1 on the segment $[-a,a]$, where $a > 0$.

Remark. If $\{\varphi_i(x)\}_1^{n+1}$ form a Chebyshev system (T-system) on $[a,b]$, then we shall regard its type as equal to $n+1$.

Theorem. In order that, for every function $f(x)$ continuous on $[a,b]$, every polynomial least deviating from it

$$p(c,x) = \sum_{i=1}^{n+1} c_i \varphi_i(x)$$

possess no fewer than $k+1$ point alternations, and, in order that for every function $f(x)$ at least one of the polynomials of least deviation from it have exactly $k+1$ point alternations, it is necessary and sufficient that the system $\{\varphi_i(x)\}_1^{n+1}$ be of type k $(0 \le k \le n+1)$ on $[a,b]$.

In what follows we shall use the criterion for polynomials of least deviation when, on $[a,b]$, the functions $\{\varphi_i(x)\}_1^{n+1}$ do not form a Chebyshev system ([1,2]), and the following lemmas.

Lemma 1. The equality is valid

$$\sum_{i=k}^{k+l-1} (-1)^{k+l-1} D\begin{pmatrix} \varphi_1, \dots, \varphi_k \\ x_1, \dots, x_{k-1}, x_i \end{pmatrix} D\begin{pmatrix} \varphi_1, \dots, \varphi_k \\ x_l, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+l-1}, x \end{pmatrix} =$$

$$= D\begin{pmatrix} \varphi_1, \dots, \varphi_k \\ x_1, \dots, x_{k-1}, x \end{pmatrix} D\begin{pmatrix} \varphi_1, \dots, \varphi_k \\ x_l, \dots, x_{k+l-1} \end{pmatrix} \qquad (k \ge 1,\ 1 \le l \le k),$$

where

$$D\begin{pmatrix} \varphi_1, \dots, \varphi_k \\ x_{t_1}, \dots, x_{t_k} \end{pmatrix} = \begin{vmatrix} \varphi_1(x_{t_1}), \dots, \varphi_k(x_{t_1}) \\ \dots \\ \varphi_1(x_{t_k}), \dots, \varphi_k(x_{t_k}) \end{vmatrix}.$$

Lemma 2. If

$$D\begin{pmatrix}\varphi_{t_1},\dots,\varphi_{t_i}\\ x_1,\dots,x_i\end{pmatrix}\neq 0,$$

then

$$(-1)^{n-i+1}D\begin{pmatrix}\varphi_1,\dots,\varphi_{n+1}\\ x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n+1},x\end{pmatrix}+$$

$$+\sum_{r=i+1}^{n+1}(-1)^{n-r+1}\frac{D\begin{pmatrix}\varphi_{t_1},\dots,\varphi_{t_i}\\ x_1,\dots,x_{j-1},x_{j+1},\dots,x_i,x_r\end{pmatrix}}{D\begin{pmatrix}\varphi_{t_1},\dots,\varphi_{t_i}\\ x_1,\dots,x_i\end{pmatrix}}\times$$

$$\times D\begin{pmatrix}\varphi_1,\dots,\varphi_{n+1}\\ x_1,\dots,x_{r-1},x_{r+1},\dots,x_{n+1},x\end{pmatrix}=$$

$$=\frac{D\begin{pmatrix}\varphi_{t_1},\dots,\varphi_{t_i}\\ x_1,\dots,x_{j-1},x_{j+1},\dots,x_i,x\end{pmatrix}}{D\begin{pmatrix}\varphi_{t_1},\dots,\varphi_{t_i}\\ x_1,\dots,x_i\end{pmatrix}}D\begin{pmatrix}\varphi_1,\dots,\varphi_{n+1}\\ x_1,\dots,x_{n+1}\end{pmatrix}$$

$$(j=1,\dots,i).$$

Lemma 3. If the system $\{\varphi_i(x)\}_1^{n+1}$ is of type k, then in the sequence $D_1^{n+1},D_2^{n+1},\dots,D_{n+2}^{n+1}$ ($D_j^{n+1}\neq 0$; $j=1,\dots,n+2$) there will be no more than $n+1-k$ changes of sign, where D_j^{n+1} ($j=1,\dots,n+2$) are obtained by deleting the j-th row from the matrix

$$\begin{Vmatrix}\varphi_1(x_1)\dots\varphi_{n+1}(x_1)\\ \dots\\ \varphi_1(x_{n+2})\dots\varphi_{n+1}(x_{n+2})\end{Vmatrix}.$$

Lemma 4. If in the sequence of complementary minors $D_1^i,\dots,D_{i+1}^i$ ($D_j^i\neq 0$, $j=1,\dots,i+1$) of the elements of one column of some determinant of order $(i+1)$ of the matrix

$$\begin{Vmatrix}\varphi_1(x_{\mu_1})\dots\varphi_{n+1}(x_{\mu_1})\\ \dots\\ \varphi_1(x_{\mu_{i+1}})\dots\varphi_{n+1}(x_{\mu_{i+1}})\end{Vmatrix}\qquad(*)$$

($\mu_1,\mu_2,\dots,\mu_{i+1}$ are the numbers of the rows appearing in condition 1) of the definition) has p changes of sign, then in the row of nonzero complementary minors D_j^i ($j=1,\dots,i+1$) of the elements of any other column of the same matrix () *there will also be p changes of sign, and if the complementary minor of at least one element of some determinant of order $(i+1)$ of the matrix* () is equal to zero, the complementary minors of all the remaining elements of the corresponding column are also equal to zero.

We outline the proof of the theorem.

Sufficiency. Let a system of type k and $p(c, x)$ deviate least from $f(x)$ on $[a, b]$. From (1) it follows that among the points of the minimal subset there will be $i + 1$ ($k \le i \le n + 1$) points $x_1, \dots, x_{i+1}$ for which: a) the matrix (*) has rank i; b) there exist $D^i_j \neq 0$ ($j = 1, \dots, i + 1$); c)
$\operatorname{sign}[p(c, x_j) - f(x_j)] = \delta \operatorname{sign}(-1)^j D^i_j$, where $\delta = \pm 1$.

Since in the row $D^i_1, \dots, D^i_{i+1}$ ($k \le i \le n + 1$) there are no more than $i - k$ changes of sign, in the row $-D^i_1, D^i_2, \dots, (-1)^{i+1} D^i_{i+1}$ there will be no fewer than k changes of sign. Consequently, at least $k + 1$ points from $\{x_i\}^{i+1}_1$ will be alternation points.

Necessity. Let every polynomial $p(c, x)$ that deviates least from $f(x)$ possess at least $k + 1$ pointwise alternations.

Suppose that the system $\{\varphi_i(x)\}^{n+1}_1$ is of type r (for definiteness, $r < k$). Then, by definition, there exists a system of points $\{x_j\}^{i+1}_1$ for which, in the corresponding row $D^i_1, \dots, D^i_{i+1}$, there will be exactly $i - r$ changes of sign.

Put $f(x_j) = \operatorname{sign}[(-1)^j D^i_j]$ ($j = 1, \dots, i + 1$), $f(a) = \operatorname{sign}(-D^i_1)$, $f(b) = \operatorname{sign}(-1)^{i+1} D^i_{i+1}$, and at the remaining points of $[a, b]$ let $f(x)$ be linear. From condition 1) of the definition we have

$$\sum_{j=1}^{i+1} (-1)^j D^i_j p(c, x_j) = 0.$$

Consequently, the least deviation

$$h(c) = |p(c, x_j) - f(x_j)| = |p(c, x_j) - [\operatorname{sign}(-1)^j D^i_j]|$$

is not less than 1. $p(c, x) \equiv 0$ deviates least from $f(x)$. For it

$$\operatorname{sign}[p(c, x_j) - f(x_j)] = -\operatorname{sign}[(-1)^j D^i_j]$$

and, consequently, the number of alternation points will be $r + 1 < k + 1$, which contradicts the assumption.

It is easy to show that any system of functions $\{\varphi_i(x)\}^{n+1}_1$ linearly independent on $[a, b]$ will be a system of some definite type k ($0 \le k \le n + 1$).

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Note: Figure translations are in progress. See original paper for figures.