ON STEADY WAVES OF FINITE AMPLITUDE CAUSED BY PRESSURE PERIODICALLY DISTRIBUTED OVER THE SURFACE OF A FLOW OF A HEAVY FLUID OF FINITE DEPTH

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Abstract

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ON STEADY WAVES OF FINITE AMPLITUDE CAUSED BY PRESSURE PERIODICALLY DISTRIBUTED OVER THE SURFACE OF A FLOW OF A HEAVY FLUID OF FINITE DEPTH

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The problem considered here was first posed and approximately solved, for a flow of infinite depth, by L. N. Sretenskii (^1). An exact solution of the problem for a flow of infinite depth and for a pressure distribution of a more general form on the surface was given in our paper (^2).

We considered an exact solution of the problem, prescribing, as in the case of infinite depth, the pressure by a certain infinite trigonometric series. We also investigated the special case when the wavelength of the prescribed pressure coincides with the length of the established free wave corresponding to the given flow velocity and constant pressure at the surface. Here we briefly set forth the results obtained by us. The main results were reported by us at the International Congress of Mathematicians in Moscow in 1966 (16–26 VIII) (^3).

Consider a plane-parallel steady motion of an ideal incompressible heavy fluid of finite constant depth h, bounded above by a free surface on which the pressure $p = p_0(x)$ is a prescribed periodic function of the horizontal coordinate x. Suppose that at the horizontal bottom the flow has a prescribed constant mean velocity c, directed from left to right. Owing to the periodically distributed pressure, the surface takes the form of a stationary periodic wave in coordinates associated with a progressive wave having velocity $-c$.

Let the wave sought and the pressure $p_0(x)$ have the same symmetry with respect to the vertical through the crest. Let us align Oy with the axis of symmetry and direct it vertically upward. As the origin of coordinates O we take the point of intersection of the axis Oy with the free surface, and direct the axis Ox to the right. We take the plane of flow xOy as the plane of the complex variable $z = x + iy$. We introduce the usual notation: φ is the velocity potential, ψ the stream function, $w = \varphi + i\psi$ the complex velocity potential, and U, V the projections of the velocity vector $\mathbf{q}$ on the coordinate axes. Then we have

$$dw/dz = -U + iV, \qquad U = -\partial\varphi/\partial x, \qquad V = -\partial\varphi/\partial y.$$

To derive the fundamental equation of the problem from the boundary condition, we first map conformally the domain occupied by one wave and representing a vertical rectangle bounded above by a wave-shaped curve onto the rectangle

$$|\varphi| \leq \tfrac{1}{2}c\lambda, \qquad 0 \leq \psi \leq \psi_0$$

(here $\psi = \psi_0$ is the flow discharge per unit time), and then this rectangle onto the interior of a circular annulus with center at zero in the plane $u = u_1 + iu_2$. It is assumed here that the wavelength λ coincides with the period of the function $p_0(x)$. As is known, the latter mapping is given by

by the formula

$$w = \frac{\lambda c}{2\pi i} \ln u. \tag{1}$$

In this case the segment $|\varphi| \leqslant \frac{1}{2}c\lambda$, corresponding to the free surface, will pass into the circumference of the outer circle of radius unity, while the segment corresponding to the bottom will pass into the circumference of the inner circle of radius

$$r_0 = \exp(-2\pi\psi_0/\varphi_0) = \exp(-2\pi h/\lambda),$$

less than unity. The annulus will have a cut along the segment $(-1, -r_0)$.

The mapping of this annulus of the u-plane onto the region of one wave in the z-plane is determined from the relation

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{f(u)}{u}. \tag{2}$$

The function $f(u)$ is represented by a Laurent series inside the annulus under consideration in the u-plane. The coefficients of this series must be real by virtue of the symmetry of the wave; moreover, the boundary condition at the bottom will also be satisfied.

Using Bernoulli' s integral for the surface, and passing in it to the variable u, we put $u = e^{i\theta}$; taking into account that on the surface $p = p_0(x)$, after differentiation with respect to θ we have

$$\frac{1}{\rho}\frac{dp_0}{dx}\frac{dx}{d\theta} = -g\frac{dy}{d\theta} - \frac{1}{2}\frac{dq^2}{d\theta}, \tag{3}$$

where θ is the angle of the radius vector in the u-plane with the axis u_1, ρ is the density, g is the acceleration due to gravity, and q is the modulus of the velocity vector.

As usual, by introducing the function [4]

$$\omega(u) = \Phi + i\tau = -i \ln f(u) \tag{4}$$

we find from (4) and (2) that, for $u = e^{i\theta}$,

$$\frac{dx}{d\theta} + i\frac{dy}{d\theta} = -\frac{\lambda}{2\pi} e^{-\tau(\theta)} (\cos\Phi + i\sin\Phi). \tag{5}$$

From formulas (4), (2), and (1) it follows that everywhere in the flow the function Φ is equal to the angle of the velocity vector q with the axis Ox, and that

$$q = c\exp(\tau). \tag{6}$$

By virtue of (5), (6), from equation (3) we obtain a differential relation; integrating it, we introduce the constant of integration

$$\mu = \frac{3g\lambda}{2\pi c^2} \exp[-3\tau(0)]. \tag{7}$$

Taking the logarithmic derivative of both sides of the indicated integral relation, we obtain an equality relating the functions $\tau(\theta)$ and $\Phi(\theta)$ on the circumference $|u| = 1$. Hence, applying the well-known Dini relation for the function $\omega(u)$, regular inside the annulus, we finally have

$$\Phi(\theta) = \mu \int_0^{2\pi} \frac{\sin\Phi + Q(\eta)\cos\Phi}{1 + \mu\int_0^{\eta}(\sin\Phi + Q\cos\Phi)\, d\eta_1} K(\eta,\theta)\, d\eta, \tag{8}$$

where

$$Q(\eta) = \frac{1}{\rho g}\frac{dp_0}{dx}, \qquad K(\eta,\theta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\eta)\varphi_n(\theta)}{\nu_n}.$$

The eigenfunctions $\varphi_n(\theta)$ and eigenvalues ν_n of the kernel $K(\eta,\theta)$ have the form

$$\varphi_n(\theta) = \sin n\theta/\sqrt{\pi}, \qquad \nu_n = 3n\ \mathrm{cth}(2\pi nh/\lambda).$$

Equation (8) is the integral equation of the problem. From this equation, for $p_0 = \mathrm{const}$, one obtains A. I. Nekrasov’ s equation (^{4}) for finite depth.

In solving equation (8) we assume that

$$\frac{1}{\rho g}\frac{dp_0}{dx} = \sum_{n=1}^{\infty} \varepsilon^n d_n \sin n\theta, \tag{9}$$

where ε is a small dimensionless positive parameter, and d_n are prescribed real numbers, the series $\sum_{n=1}^{\infty} \varepsilon^n d_n$ converging for $0 < \varepsilon < \varepsilon_0$. Recall that in the original problem p_0 is a prescribed periodic function of x. In our paper (^{2}) we indicated the relation that holds between the original problem and the problem under condition (9).

The parametric equation of the wave profile is obtained from (5) in the form

$$x = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta. \tag{10}$$

Formulas (10) show that, in solving the problem, in addition to $\Phi(\theta)$, it is necessary to find also $\tau(\theta)$. Both of these functions are represented by trigonometric series

$$-\tau(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad \Phi(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{11}$$

From the Laurent expansion of the function $\ln f(u) = i\omega(u)$ we obtain relations between the coefficients of these series,

$$A_n = \frac{\nu_n}{3n} B_n \qquad (n = 1, 2, \ldots). \tag{12}$$

Thus, knowing B_n, we find all A_n, except A_0.

We transform formula (7). Setting

$$\mu_0 = 3g\lambda/2\pi c^2, \tag{13}$$

from equations (11), (13), and (7) we have

$$\mu = \mu_0 \exp\left[3\left(A_0 + \sum_{n=1}^{\infty} A_n\right)\right]. \tag{14}$$

Putting $\theta = 2\pi$ in the right-hand side of the first formula (10), we must obtain $-\lambda$ in its left-hand side, since then x decreases by λ. In this way we obtain an equation for determining A_0,

$$\exp(-A_0) = \frac{1}{2\pi} \int_0^{2\pi} \exp[-\tau(\theta) - A_0] \cos\Phi(\theta)\, d\theta; \tag{15}$$

here $-\tau(\theta) - A_0$, by virtue of (11), does not contain A_0.

Putting

$$\Psi(\theta) = \left[1 + \mu \int_0^{\theta} (\sin\Phi + Q(\eta)\cos\Phi)\, d\eta\right]^{-1}, \tag{16}$$

we reduce, as in the case of infinite depth (^{2}), equation (8) to an equivalent system of two equations with unknown functions $\Phi(\theta)$ and $\Psi(\theta)$.

Thus, the problem has been reduced to the determination of the functions $\Phi(\theta, \varepsilon)$ and $\Psi(\theta, \varepsilon)$ from the indicated equivalent system, of the parameter $\mu(\varepsilon)$ from (14), and of the coefficient $A_0(\varepsilon)$ from (15). In solving it, one has to consider two cases: in the first case $\mu_0 \neq \nu_n$, in the second $\mu_0 = \nu_n$. In the first case the solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, $\mu(\varepsilon)$, $A_0(\varepsilon)$ is constructed in the form of series in integral powers of the parameter ε. In the second case, as an example, we considered the value $\mu_0 = \nu_1$; here the solution is obtained in the form of series in powers of $\varepsilon^{1/3}$. In both cases, by the Lyapunov-Schmidt methods (^{5}), we prove that these series converge absolutely and uniformly for $0 \leq \theta \leq 2\pi$ and small values $|\varepsilon| < \varepsilon_1 \leq \varepsilon_0$, and that they give the unique solution of the problem which is small relative to ε and continuous in θ.

The wave profile in parametric form is given by equations (10). After substituting into these equations the found $\tau(\theta, \varepsilon)$ and $\Phi(\theta, \varepsilon)$ and eliminating θ from them, we obtain the equation of the profile in the form $y = y(x, \varepsilon)$.

We give the profile equations in both cases, approximate up to terms of second order. The terms of third order computed by us are not given because of their cumbersomeness.

In the case $\mu_0 \neq \nu_n$,

$$y = (x, \varepsilon) = \frac{1}{k}\left\{\varepsilon C_{11}(\cos kx - 1) + \frac{\varepsilon^2}{2}\left[\frac{\nu_1}{6} C_{11}^2 - \frac{1}{\nu_2 - \mu_0}\left(\frac{\nu_1^2}{2} C_{11}^2 + \mu_0 d_2\right)\right](1 - \cos 2kx)\right\},$$

where $C_{11} = d_1 \mu_0 / \nu_1 - \mu_0$, $k = 2\pi/\lambda$.

In the case $\mu_0 = \nu_1$,

$$y(x, \varepsilon) = \frac{1}{k}\left[\varepsilon^{1/3}(d_1\beta)^{1/3}(\cos kx - 1) + \frac{\varepsilon^{2/3}}{2}(d_1\beta)^{2/3}\frac{\nu_1^2(\nu_2 - 4\nu_1)}{6(\nu_2 - \nu_1)}(1 - \cos 2kx)\right],$$

where

$$\beta = \frac{24(\nu_2 - \nu_1)}{(15 - 8\nu_1^2)(\nu_2 - \nu_1) + 3\nu_1^2(2\nu_2 - \nu_1)}.$$

By the condition of the problem, the origin of coordinates is placed at the wave crest. Therefore, from analysis of the principal terms in the formulas for $y = y(x, \varepsilon)$, and assuming $\nu_1 < \mu_0 < \nu_2$, we conclude that one must take $d_1 < 0$.

Let us also note that $\mu_0 = \nu_1$ is the special case which was mentioned at the beginning of the article.

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Note: Figure translations are in progress. See original paper for figures.

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