PALEY SINGULARITIES AND THE RUDIN-CARLESON INTERPOLATION THEOREMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract

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S. A. VINOGRADOV

PALEY SINGULARITIES AND THE RUDIN-CARLESON INTERPOLATION THEOREMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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As is well known, the Fourier coefficients of a function continuous on the circle $\{e^{it} : |t| \leqslant \pi\}$ may tend to zero very slowly even in the case when this function admits a continuous extension to a function holomorphic in the open unit disk

$$D = \{re^{it} : 0 \leqslant r < 1,\ |t| \leqslant \pi\}.$$

This circumstance manifests itself, in particular, in the so-called Paley singularities, which were the subject of investigation in papers ([1,5−8]). In this connection let us also mention Banach' s theorem ([2]), which asserts, roughly speaking, that the Fourier coefficients of a continuous function, taken with indices forming a lacunary sequence, are indistinguishable from the Fourier coefficients of an arbitrary function of class L^2. On the other hand, Rudin ([3]) and Carleson ([4]) showed that from certain infinite subsets of the closed unit disk $\overline{D}$, every continuous function can be extended to a function holomorphic in D and continuous in $\overline{D}$.

In the present note we formulate assertions which generalize and refine some results of the papers of R. Paley ([1]), S. B. Stechkin ([5]), V. P. Khavin ([6]), and A. S. Makhmudov ([8]) on singularities of Paley type (Theorem 1 and its corollaries). It turns out that the singularities studied by these authors already occur for functions analytic in the whole complex plane, except for the point $z = 1$, and continuous in the closed disk $\overline{D}$. In addition, a theorem is formulated which simultaneously generalizes the above-mentioned theorems of Banach and Rudin–Carleson (Theorem 3). In Theorems 4 and 5 we are concerned with certain singularities of power series uniformly convergent in $\overline{D}$.

Let us introduce some notation.

1°. $\hat{C}$ is the extended complex plane,

$$D = \{z \in \hat{C} : |z| < 1\}, \qquad \partial D = \{z \in C : |z| = 1\}, \qquad \overline{D} = D \cup \partial D.$$

2°. G_α is a domain in the extended complex plane such that:
a) $D \subset G_\alpha$, the point at infinity belongs to the domain G_α; b) the boundary ∂G_α of the domain G_α is a closed rectifiable Jordan curve containing the arc of the circle

$$\Gamma_\alpha = \{e^{i\theta} : |\theta| < \alpha\},$$

where $0 < \alpha < \pi$.

3°. $A(G_\alpha)$ is the space of all functions f holomorphic in the domain G_α and continuous in $\overline{G_\alpha}$ ($\overline{G_\alpha}$ is the closure of G_α), with norm

$$\|f\|_{A(G_\alpha)} = \sup_{z \in G_\alpha} |f(z)|.$$

4°. T is a closed set lying on the unit circle ∂D ($T \neq \partial D$), and $A_C(T)$ is the space of all functions holomorphic everywhere outside the set T (including the point at infinity) and continuous in the closed disk $\overline{D}$. In particular, if $T = \{1\}$, then $A_0(\{1\})$ is the space of all functions holomorphic everywhere except $z = 1$ and continuous in $\overline{D}$.

5°. E is an arbitrary infinite set consisting of nonnegative integers,

$$E_n = \{k \in E : k \leqslant n\},$$

where n is a natural number.

6°. $d = \{d_k\}_{k \in E}$ is a family consisting of positive numbers.

7°.

$$M_n(p, d) = \sup_{\|f\|_{A(G_\alpha)} \leqslant 1} \left(\sum_{k \in E_n} |\hat{f}(k)|^p d_k^{2-p} \right)^{1/p}, \quad 0 < p < 2, \quad n = 1, 2, \dots,$$

where

$$\hat{f}(k) = \frac{1}{2\pi i} \int_{|z|=1} f(z) z^{-(k+1)}\, dz, \quad k = 0, 1, 2, \dots .$$

8°. $l^2(E)$ is the space of all families of complex numbers $x = \{x_k\}_{k \in E}$ such that the norm

$$\|x\|_2 = \left(\sum_{k \in E} |x_k|^2 \right)^{1/2}$$

is finite.

9°. R is the linear operator acting from $A_C(T)$ into $l^2(E)$ as follows: $Rf = \{\hat{f}(k)\}_{k \in E}$ ($\hat{f}(k)$ is defined in 7°).

10°. S is the restriction operator, defined on the space of functions $A_C(T)$ on the set Q ($Q \subset \bar{D}$), i.e.

$$(Sf)(z) = f(z), \quad z \in Q, \quad f \in A_C(T).$$

Theorem 1. Let G_α be the domain defined in 2°. Then there exists a finite positive constant $B(\alpha, p)$ such that

$$B(\alpha,p)\left(\sum_{k\in E_n} d_k^2\right)^{1/p-1/2} \leqslant M_n(p,d) \leqslant \left(\sum_{k\in E_n} d_k^2\right)^{1/p-1/2}, \quad p\in(0,2), \quad n=1,2,\dots .$$

This theorem is a generalization of the well-known Paley theorem (see ([1])).

Corollary 1. Let the family $d = \{d_k\}_{k\in E}$ satisfy the condition

$$\sum_{k\in E} d_k^2 = +\infty.$$

Then there exists a function f, holomorphic everywhere except at the point $z = 1$, continuous in $\bar{D}$ (i.e. $f \in A_C(\{1\})$) and such that

$$\sum_{k\in E} |\hat{f}(k)|^p d_k^{2-p} = +\infty$$

for all $p \in (0, 2)$.

Corollary 2. Let $\{t_k\}_{k=0}^{\infty}$ be a sequence of nonnegative numbers such that

$$\sum_{k=0}^{\infty} t_k^r = +\infty$$

for every $r > 0$. Then there exists a function f such that $f \in A_C(\{1\})$ and

$$\sum_{k=0}^{\infty} t_k |\hat{f}(k)|^{2-\varepsilon} = +\infty \quad \text{for all } \varepsilon\in(0,2).$$

Corollaries 1 and 2 are a generalization of certain results of V. P. Khavin contained in ([6]).

Remark. All results from ([7,8]) concerning the space of all functions holomorphic outside the ray $[1, +\infty)$ and continuous in $\bar{D}$ remain valid if this space is replaced by the space $A_C(\{1\})$.

Definition. We shall say that the set E satisfies the Hadamard condition (or is lacunary) if

$$\gamma = \inf_{\substack{0<k<m\\ k,m\in E}} m/k > 1.$$

Theorem 2. Let the set E satisfy the Hadamard condition. Then

$$R(A_C(\{1\})) = l^2(E).$$

This theorem is a strengthening of the well-known theorem of Banach ([2]).

Remark. The Hadamard condition can be considerably weakened.

Theorem 3. Let the sets T and Q, appearing in 4° and 10°, coincide ($Q = T$), and let $\operatorname{mes} T = 0$ (mes denotes Lebesgue measure on

circle ∂D, and the set E satisfies Adamyan' s condition. Then for every function ψ, continuous on the set T, and for every family $x = \{x_k\}_{k \in E} \in l^2(E)$, there exists a function $f \in A_C(T)$ such that $Sf = \psi$ and $Rf = x$.

In particular, this theorem is a strengthening of the well-known theorem of Rudin-Carleson (see ([3],[4])).

Denote by $AU(\{1\})$ the space of all functions f, holomorphic everywhere except at the point $z = 1$ and having a Maclaurin series uniformly convergent on the unit circle ∂D.

Theorem 4. Let $\{d_k\}_{k=0}^{\infty}$ be a sequence of nonnegative numbers and $p \in (0, 2)$. Then, if

$$\frac{1}{\ln(n+2)} \left(\sum_{k=0}^{n} d_k^2 \right)^{1/p - 1/2} \longrightarrow +\infty, \qquad n \to \infty,$$

there exists a function f such that:

1. $f \in AU(\{1\})$.
2. $$\sum_{k=0}^{\infty} |\hat{f}(k)|^p d_k^{2-p} = +\infty.$$

Corollary. There exists a function f such that:

1. $f \in AU(\{1\})$.
2. $$\sum_{k=0}^{\infty} |\hat{f}(k)|^{2-\varepsilon} = +\infty \quad \text{for all } \varepsilon \in (0, 2).$$

Denote by the symbol $AU_0(\{1\})$ the space of all functions f such that $f \in AU(\{1\})$ and

$$\hat{f}(n) = o\left(\frac{1}{n}\right), \qquad n \to \infty$$

($\hat{f}(n)$ is defined in 7°), and by the symbol $C(Q)$ the space of all functions uniformly continuous on the set Q ($Q \subset D$).

Theorem 5. Let the set Q ($Q \subset D$) satisfy the condition: there exists a number $\lambda \in [1, +\infty)$ such that

$$|1-\eta| \le \lambda(1-|\eta|) \tag{I}$$

for all $\eta \in Q$.

Then the following assertions are equivalent:

1. $S(AU_0(\{1\})) = C(Q)$.
2. The set Q satisfies Carleson' s condition (see ([10])), i.e.

$$\delta = \inf_{\xi \in Q} \prod_{\eta \in Q_\xi} \left| \frac{\xi - \eta}{1 - \xi\bar{\eta}} \right| > 0,$$

where $Q_\xi = \{\eta \in Q : \eta \ne \xi\}$.

Remark. It is interesting to compare the assertion of Theorem 5 with Theorem 1 from ([9]), which is formulated as follows:

Let l^1_A denote the space of all functions f such that

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k, \qquad z \in D, \qquad \sum_{k=0}^{\infty} |\hat{f}(k)| < +\infty;$$

S is the restriction operator:

$$(Sf)(z) = f(z), \qquad z \in Q, \qquad f \in l^1_A.$$

Then $S(l^1_A) = C(Q)$ if and only if Q is a finite set.

At the same time, there exist infinite sets satisfying both condition (I) and Carleson' s condition (see ([10])). For example, $Q = \{1 - 2^{-k} : k = 1, 2, ...\}$.

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.