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Abstract

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MATHEMATICS

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ON A CRITERION FOR THE STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH PERIODIC OPERATOR COEFFICIENTS IN A BANACH SPACE

(Presented by Academician I. N. Vekua on 11 III 1968)

Consider in a Banach space X the equation

$$dx/dt = A(t)x, \qquad -\infty < t < \infty, \tag{1}$$

where $A(t)$ is a T-periodic operator-function whose values are linear operators in the space X. At present many results are known (see the bibliography in ([1])) concerning theorems on the existence and uniqueness of an ordinary or generalized solution $x(t)$ of equation (1), established under certain conditions imposed on the function $A(t)$. In what follows, by $x(t)$ we shall everywhere mean an ordinary or generalized solution of equation (1). Denote by R the ring of all linear bounded operators acting in the space X.

An operator-function $U(t)$ is called a resolving one if, for any solution $x(t)$ of equation (1),

$$x(t) = U(t)x(0). \tag{2}$$

In relation (2), $U(t) \in R$ for every $t \in (-\infty, \infty)$. In those cases when an arbitrary element of X is taken as the initial value $x(0)$ of the solution $x(t)$, the operator $U(t)$ maps the space X one-to-one onto itself and, by the Banach theorem, is continuously invertible.

Equation (1) is called stable if all its solutions $x(t)$ are bounded on the axis $(-\infty, \infty)$.

By virtue of the uniqueness of the solution $x(t)$, it follows from (2) that

$$U(t+T) = U(T)U(t). \tag{3}$$

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The operator $U(T)$ is called the monodromy operator of equation (1). Put $r_t = t - nT$ for any t in $(-\infty, \infty)$, where $n = 0, \pm 1, \pm 2, \ldots.$ Then from (2) and (3) we obtain

$$x(t) = U^n(T)U(r_t)x(0). \tag{4}$$

The operator-function $U(t)$ is bounded on the segment $[0, T]$, and therefore from (4) and the Banach-Steinhaus theorem it follows that boundedness of any solution $x(t)$ on the entire axis $(-\infty, \infty)$ is equivalent to the relation

$$\|U^n(T)\| \leq C, \qquad C = \text{const}; \qquad n = 0, \pm 1, \pm 2, \ldots \tag{5}$$

Definition. An operator U from R satisfying estimate (5) is called stable. Thus, the stability of equation (1) is equivalent to the stability of the monodromy operator $U(T)$ of this equation.

An operator V from R mapping all of X onto X and such that

$$\|Vx\| = \|x\|, \tag{6}$$

is called unitary.

In the case when the space X is Hilbert, a stable operator U, by the theorem of B. Sz.-Nagy ([2]), is similar to a unitary operator V in the ordinary sense. The following theorem is a generalization of the theorem of B. Sz.-Nagy.

Theorem 1. *A stable operator U in the space X is similar to a unitary operator V acting in this space.*

Proof. The group $G = \{U^n\}_{-\infty}^{\infty}$ is commutative and bounded by the constant C, and therefore, by a well-known theorem of A. A. Markov ([3]), in the space of all bounded numerical functions on G there exists a linear functional $M(\varphi)$ with the properties:

$$\begin{aligned} &1) \quad M(\varphi(AB)) = M(\varphi(A)), \quad A, B \in G; \\ &2) \quad M(\varphi(A)) \geq 0 \quad \text{if } \varphi(A) \geq 0; \\ &3) \quad M(1) = 1. \end{aligned}$$

Let X^* be the space conjugate to the space X. By the Hahn–Banach theorem, for any element x of X there exists a functional x^* from X^* such that $x^*x = \|x\|$. Thus a mapping W from X into X^* is defined, and we can define on the space X the form

$$[x, y] = Wyx = y^*x. \tag{7}$$

The form $[x, y]$ is linear in the first argument; the quadratic form $[x, x]$ is positive definite, and therefore the form $[x, y]$ is called a semiscalar product.

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The expression $\varphi_{x,y}(A) = [Ax, Ay]$, where $A \in G$, is a function on the group G. It is obvious that the function $\varphi_{x,y}(A)$ is bounded on G. In particular, for $A = I$ we have

$$\varphi_{x,y}(I) = [x, y]. \tag{8}$$

Take as the new norm $\|x\|_1$ of the element x of the space X the quantity $M([Ax, Ax])$. By property 1), for any B from G we obtain that

$$M([ABx, ABx]) = M(\varphi_{x,x}(AB)) = M(\varphi_{x,x}(A)) = M([Ax, Ax]).$$

Thus, for all elements U^n of the group G we have the relation

$$\|U^n x\|_1 = \|x\|_1. \tag{9}$$

For $n = 1$, from (9) we obtain

$$\|Ux\|_1 = \|x\|_1. \tag{10}$$

The topological equivalence of the norms $\|x\|$ and $\|x\|_1$ is obvious from the definition of the norm $\|x\|_1$. Theorem 1 is proved.

Corollary 1. *For stability of equation (1) it is necessary and sufficient that the monodromy operator $U(T)$ of this equation be similar to a unitary operator.*

We now indicate the class of stable unitary operators in the space X.

In ([4]) the concept was introduced of an operator of spectral type A, decomposed into the sum of an operator of scalar type

$$S = \int_{\sigma(S)} \lambda \, dE_\lambda$$

and a radical operator N:

$$A = S + N. \tag{11}$$

It is known ([5]) that a spectral operator A is an operator of scalar type with spectrum lying on the unit circle if and only if estimate (5) is satisfied for the operator A. Thus, the set $Y(S)$ of all scalar-type operators S from R with spectrum on the unit circle is contained in the set Y of all stable operators from R.

Theorem 2. If $U(T) \in Y(S)$, then equation (1) is stable.

If X is a Hilbert space, then every operator in Y belongs to the set $Y(S)$, and consequently in this case

$$Y(S) = Y. \tag{12}$$

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Theorem 3. In a Hilbert space X, equation (1) is stable if and only if the monodromy operator $U(T) \in Y(S)$.

In the general case the relation (12) does not hold; moreover, in ([6]) examples are given of unitary operators in X which are not even operators of spectral type.

Thus, in the general case the condition $U(T) \in Y(S)$ is not necessary for the stability of equation (1). All known theorems ([1]) guaranteeing the existence of a solution $x(t)$ of equation (1) contain, along with other conditions, the requirement that the spectrum of the operator $A(t)$, for each $t \in (-\infty, \infty)$, be contained in a fixed sector S_z of the plane, where for all $z \in S_z$, $\pi/2 < \arg z < 3\pi/2$. Thus there naturally arises the question of finding simple conditions under which the spectrum $\sigma(A(t))$ of the operator $A(t)$ will be located in the above-mentioned sector S_z. We shall now establish one such condition.

In applied problems one often encounters ([7,8]) the equation

$$d^2u/dt^2 + P(t)u = 0, \qquad -\infty < t < \infty, \tag{13}$$

where $P(t)$ is a T-periodic operator-function whose values are self-adjoint operators in a Hilbert space $\mathfrak{H}$. Setting $du/dt = u_1,\ x = (u, u_1)$, we reduce equation (13) to a system of first-order equations, written in matrix notation in the space $\mathfrak{H}_0 = \mathfrak{H} \oplus \mathfrak{H}$ in the form (1), where

$$A(t) = \begin{pmatrix} 0 & I \\ -P(t) & 0 \end{pmatrix}. \tag{14}$$

Denote by $\rho(A(t))$ the resolvent set of the operator $A(t)$.

Theorem 4. If $\rho(A(t)) = \emptyset$, and for all $x, y \neq \{0\}$ from the domain of definition $\mathcal{D}(P(t))$ of the operator $P(t)$ it follows from $\mathrm{Re}(x, y) = 0$ that $(P(t)x, y) \neq (y, y)$, then the spectrum $\sigma(A(t))$ is real.

Proof. Introduce in the space $\mathfrak{H}_0$ an operator J, whose matrix representation in the decomposition $\mathfrak{H}_0 = \mathfrak{H} \oplus \mathfrak{H}$ has the form

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{15}$$

Consider in $\mathfrak{H}_0$ the form

$$[x, y] = (Jx, y), \tag{16}$$

where $(x, y) = (x_1, y_1) + (x_2, y_2)$, $x = \{x_1, x_2\}$, $y = \{y_1, y_2\} \in \mathfrak{H}_0$. The quadratic form $[x, x]$ on $\mathfrak{H}_0$ assumes the values $[x, x] > 0$, $[x, x] < 0$, $[x, x] = 0$, $\{0\} \neq x \in \mathfrak{H}_0$.

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A Hilbert space $\mathfrak{H}_0$, equipped, in addition to the usual scalar product (x, y), with the indefinite scalar product $[x, y]$, is called a space with an indefinite metric. The notions of J-self-adjoint, J-isometric, and J-unitary operators in $\mathfrak{H}_0$ with respect to the form $[x, y]$ are introduced in the usual way. By definition, an operator A in $\mathfrak{H}_0$ with dense domain $D(A)$ is J-self-adjoint if

$$(JA)^* = JA. \tag{17}$$

From (14) and (15) we obtain

$$JA(t) = \begin{pmatrix} -P(t) & 0 \\ 0 & I \end{pmatrix}; \tag{18}$$

Thus, for every $t \in (-\infty, \infty)$ the operator $A(t)$ is a J-self-adjoint operator in $\mathfrak{H}_0$. A J-self-adjoint operator A is said to belong to the class of Pesonen [(9)] ($A \in \Pi$) if from $[x, x] = 0$ and $[Ax, x] = 0$ ($x \in \mathfrak{D}(A)$) it follows that $x = \{0\}$. Denote by $\Pi^+(\Pi^-)$ the set of operators in Π for which from $(x, x) = 0$, $\{0\} \neq x \in \mathfrak{D}(A)$, it follows that $[Ax, x] > 0$ ($[Ax, x] < 0$). It is known [(10)] that $\Pi = \Pi^+ \cup \Pi^-$. Let $\{0\} = z \in \mathfrak{D}(A(t))$. In view of the decomposition $\mathfrak{H}_0 = \mathfrak{H} \oplus \mathfrak{H}$, $z = x + y$, $x, y \in \mathfrak{H}$. Obviously,

$$[z, z] = (x, y) + (y, x). \tag{19}$$

It follows from (19) that $[z, z] = 2\,\mathrm{Re}(x, y)$, and therefore $[z, z] = 0$ if and only if $\mathrm{Re}(x, y) = 0$.

Next we have

$$A(t)z = \{y, -P(t)x\},$$

$$[A(t)z, z] = (y, y) - (P(t)x, x).$$

By the condition of the theorem, if $[z, z] = 0$, $z \neq \{0\}$, then $[A(t)z, z] \neq 0$. Thus $A(t) \in \Pi$. Since $\Pi = \Pi^+ \cup \Pi^-$, either $A(t) \in \Pi^+$, or $A(t) \in \Pi^-$. Now from Theorem A.I of [(11)] it follows that the spectrum $\sigma(A(t))$ of the operator $A(t)$ is real. In applications $P(t)$, $t \in (-\infty, \infty)$, is a positive definite self-adjoint operator in $\mathfrak{H}$.

Theorem 5. *If, for all $\{0\} \neq x, y \in \mathfrak{D}(P(t))$, from $\mathrm{Re}(x, y) = 0$ it follows that $(P(t)x, x) \neq (y, y)$, and, moreover,*

$$\mathrm{Re}[-(P(t)x, y) + \overline{(x, y)}] \leq 0,$$

then the spectrum of the operator $A(t)$ lies on the negative half-axis.

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Indeed, the condition

$$\mathrm{Re}[-(P(t)x, y) + \overline{(x, y)}] \leq 0$$

means that $\mathrm{Re}[A(t)z, z] \leq 0$ for all $z \in \mathfrak{D}(A(t))$, and therefore the spectrum $\sigma(A(t))$ lies in the left half-plane. Thus, $\rho(A(t)) \neq \emptyset$, and this, together with the condition that from $\mathrm{Re}(x, y) = 0$ it follows that $(P(t)x, x) \neq (y, y)$, ensures the reality of the spectrum $\sigma(A(t))$.

It is easy to see that if the conditions of Theorem 5 are fulfilled, then for every fixed t from $(-\infty, \infty)$ the operator $A(t)$ generates a strongly continuous one-parameter group of contracting operators.

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Note: Figure translations are in progress. See original paper for figures.

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