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Abstract

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MATHEMATICS

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ON THE THEORY OF THE RIEMANN PROBLEM IN THE CLASS L_p

(Presented by Academician N. I. Muskhelishvili on VII 7, 1967)

The formulation of the Riemann boundary-value problem ([1], [2]) has been generalized in various directions. B. V. Khvedelidze ([3]) gave a complete solution of this problem in the following formulation. Find a function $f(z)$, piecewise holomorphic in the plane cut along a closed Lyapunov curve Γ, having finite order at infinity, satisfying the boundary condition

$$f^+(t) = G(t)f^-(t) + g(t), \qquad t \in \Gamma, \tag{1}$$

which relates the limiting values of the function $f(z)$ on the curve Γ. The function $G(t)$ satisfies on Γ the Hölder condition (condition H) with some exponent α $(0 < \alpha \leqslant 1)$, $g(t) \subset L_p$ $(p > 1)$. The solution is sought in the class of functions representable by an integral of Cauchy type with summable density. It turned out that in this class of functions the dependence of the number of solutions on the index of the coefficient $G(t)$ and the formulas representing the solution of the problem remain the same as in the case when $G(t)$, $g(t)$ satisfy condition H.

The question arises whether all solutions of problem (1) are exhausted by solutions representable by an integral of Cauchy type. Of course, this question will make sense if we indicate what class of functions we admit as solutions of the problem. In the present note it is shown that the solutions representable by an integral of Cauchy type, which we shall call classical, exhaust all solutions of problem (1), if the class of limiting values of the sought functions is enlarged to a certain space of generalized functions. This class includes piecewise holomorphic functions representable by an integral of Cauchy type with summable density, but this class also contains such "bad" functions as, for example, the function

$$\delta(x_0 - z) = \frac{1}{2\pi i}\frac{1}{x_0 - z}, \qquad x_0 \in \Gamma,$$

whose limiting values on the curve Γ, in the sense of the theory of generalized functions, are given by the formula

$$\delta^{\pm}(x_0 - t) = \pm\frac{1}{2}\delta(x_0 - t) + \frac{1}{2\pi i}\frac{1}{x_0 - t}\ \ * . \tag{2}$$

Our result, roughly speaking, guarantees that functions having limiting values of the form (2), and similar ones, cannot be solutions of problem (1).

§ 1. Denote by H_α the Banach space of functions $\varphi(t)$, defined on the simple closed Lyapunov curve Γ, satisfying the Hölder condition with exponent $\alpha(0 < \alpha < 1)$ with the usual norm. In what follows we shall call the space H_α the basic space, and its elements basic functions. Linear continuous functionals $(f, \varphi(t))$ will be called generalized functions (g.f.). By H_α^+ we denote the subspace,

* By $\delta(t - t_0)$ is denoted the Dirac delta function.

formed by those functions from H_α which are boundary values of functions analytic in D^+ (the interior of Γ) and continuous in the closed domain $\overline{D^+}$. Similarly, by H_α^- we shall denote the subspace formed by functions from $\bar{H}_\alpha$ that are boundary values of functions analytic in D^- (the exterior of Γ) and vanishing at infinity. In the space of generalized functions, to the subspaces H_α^+, H_α^- there correspond subspaces $H_\alpha'^+$ and $H_\alpha'^-$ according to the following law. A generalized function f^+ belongs to H'^+ if $(f^+, \varphi^+(t)) = 0$ for all $\varphi^+(t) \in H_\alpha^+$. A generalized function $f^- \in H_\alpha'^-$ if $(f^-, \varphi^-(t)) = 0$ for all $\varphi^-(t) \in H_\alpha^-$. If $f^+(z)$ is an analytic function belonging to the class E_1 in the domain D^+, for example, the classical solution of problem (1), then $f^+(t) \in H_\alpha'^+$.

Let

$$S\varphi = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma.$$

It is known that the operator S in the space H_α is bounded.

§ 2. By the Riemann problem in the space of generalized functions we shall mean the problem of finding generalized functions $f^+ \in H_\alpha'^+$ and $f^- \in H_\alpha'^-$ satisfying the condition

$$f^+ = G(t) f^- + g, \quad t \in \Gamma, \tag{3}$$

where $G(t) \in H_\alpha$, $G(t) \neq 0$, $g \in H_\alpha'$, Γ is a Lyapunov contour. We shall show that the dependence of the number of solutions of this problem on the index of the coefficient $G(t)$ will be the same as in the case when $g(t) \in L_p$. Hence, in particular, the result formulated above on the absence in problem (1) of solutions other than those representable by an integral of Cauchy type will follow if the class of boundary values of the sought functions is extended to the space of functionals H_α'.

A solution of the Riemann problem in the space of generalized functions is given, for example, in ([4]); however, in that work different, more restrictive assumptions are adopted concerning the properties of the contour and of the basic space. Nevertheless, the method of ([4]) also remains valid in the present case, since it is based on two facts that also hold in the case of the space H_α, namely, on the continuous dependence of the solution of the problem

$$\varphi^+(t) = [G(t)]^{-1}\varphi^-(t) + \varphi(t), \quad \varphi(t) \in H_\alpha, \tag{4}$$

adjoint to problem (3), on the free term, which follows from the boundedness of the singular operator S in the space H_α, and on the possibility of constructing in the space H_α a system of functions biorthogonal to any prescribed finite system of fundamental functions $((^1), pp.567—570)$.

Arguing in the same way as in the paper ([4]), we obtain, for $\chi = \operatorname{ind} G(t) = 0$,

$$(f^+, \varphi(t)) = (g, \varphi^+(t) - \varphi(t)), \quad (f^- \varphi(t)) = -(g, [\varphi(t)/G(t)]^+),$$

where $\varphi(t)$ is an arbitrary basic function; $\varphi^+(t)$ is the boundary value of the solution of problem (4), and $[\varphi(t)/G(t)]^+ = \psi^+(t)$ is the boundary value of the solution of the problem $\psi^+(t) = [G(t)]^{-1}[\psi^-(t) + \varphi(t)]$. It is easy to verify that the functionals found satisfy the boundary condition (3). Their continuity follows from the fact that $\|\varphi^\pm(t)\|_\alpha \to 0$ when $\|\varphi(t)\|_\alpha \to 0$. From the method of finding the solution there follows its uniqueness. Hence it follows that, for $\chi = 0$, problem (1) can have no solution other than the classical one.

If $\chi < 0$, then the solution of problem (4) will contain a linear combination, with arbitrary coefficients, of the solutions of the corresponding homogeneous problem. From the relation $(f^+, \varphi(t)) = (g, \varphi^+(t) - \varphi(t))$ it is seen that the functional $(f^+, \varphi(t))$ will be defined and continuous if and only if the conditions

$$(g, \varphi_k^+(t)) = 0, \quad k = 1, 2, \dots, |\chi|,$$

are satisfied, where $\varphi_k^+(t) = t^{k-1}/X^+(t)$ are the solutions of the homogeneous problem $\varphi^+(t) = [G(t)]^{-1}\varphi^-(t)$.*

* Here and below, by $X^\pm(t)$ we denote the boundary values of the canonical function $X(z)$ of the homogeneous Riemann problem $F^+(t) = G(t)F^-(t)$.

These same conditions also ensure the continuity of the functional $(f^-, \varphi(t))$. Thus, they will be solvability conditions for problem (3). If $g \in L_p$, then these conditions obviously coincide with the solvability conditions for problem (1). From the uniqueness of the solution of problem (3) under the solvability conditions it follows that problem (1), for $\chi < 0$, in the case of its solvability has no solutions other than the classical one.

For $\chi > 0$ the adjoint problem is solvable only when conditions of the form

$$\int_\Gamma \varphi(t)h_k(t)\,dt = 0, \qquad k = 1, 2, \dots, \chi, \tag{5}$$

are fulfilled, where $h_k(t) = t^{k-1}X^+(t) \in H_\alpha$. Thus, in the present case the required functionals can be determined only on the subspace H^0_α of the basic space H_α, consisting of the basic functions satisfying conditions (5). To continue the functionals $(f^\pm, \varphi(t))$ to the whole basic space we use the theorem on extension of functionals (([4], p. 279). The proof of this theorem is based on the possibility of constructing in the basic space a system of functions biorthogonal to the system $h_k(t)$, $k = 1, 2, \dots, \chi$. For this reason it remains valid in the case of the space H_α. Applying the theorem, we obtain

$$(f^+, \Phi(t)) = (X^+(t)\{[g/X^+(t)]^+ + P_{\chi-1}(t)\}, \Phi(t)), \tag{6}$$

$$(f^-, \Phi(t)) = (X^-(t)\{[g/X^+(t)]^- + P_{\chi-1}(t)\}, \Phi(t)). \tag{7}$$

In (6) and (7), $\Phi(t)$ is an arbitrary basic function; $P_{\chi-1}(t)$ is a polynomial of degree $\chi - 1$ with arbitrary coefficients; $\Psi^\pm(t) = [g/X^+(t)]^\pm$ is the solution of the problem $\Psi^+ - \Psi^- = g/X^+(t)$. From formulas (6), (7) we conclude that in this case as well all solutions of problem (1) are exhausted by the classical ones.

The result obtained remains valid also in the case when $g(t)$ belongs to the class of functions summable to the power p with weight

$$\prod_{k=1}^{m_1}(t - t_k)^{-\beta_k} \prod_{k=m_1+1}^{m}(t - t_k)^{\beta_k(p-1)}, \qquad t_k \in \Gamma,\ m \geqslant 1,\ m_1 \leqslant m,\ 0 < \beta_k < 1,$$

since the boundary values of the solutions of this problem, representable by an integral of Cauchy type (see ([3])), fall into this same class, which is entirely contained in the space of functionals H'_α. Indeed, in a neighborhood of any of the points t_k $(k > m_1)$, $g \in L_r$ for

$$r < \frac{1}{p} + \beta_k \frac{1}{q}, \qquad \text{where } q = \frac{p}{p-1}.$$

§ 3. Let now, in the boundary condition (1), the coefficient $G(t)$ be different from zero and have on Γ discontinuities of the first kind at the points t_k, $k = 1, 2, \dots, n$, satisfying on the closed arcs (t_k, t_{k+1}) the Hölder condition with exponent α $(0 < \alpha < 1)$, while $g(t)$ is a function summable to the power p $(p > 1)$ with some weight. Suppose that t_k, $k = 1, 2, \dots, m$, are nonexceptional points of discontinuity of the coefficient $G(t)$ (([1], p. 330) and set, as usual,

$$\lambda_k = \frac{1}{2\pi i} \ln \frac{G(t_k - 0)}{G(t_k + 0)} = \frac{\theta_k}{2\pi} - m_k - \frac{i}{2\pi} \ln r_k,$$

where m_k is an integer whose choice determines the branch of the logarithm. Put $\alpha_k = \operatorname{Re} \lambda_k = \theta_k/2\pi - m_k$, and choose m_k so that $0 < \alpha_k < 1$ $(k \leqslant m)$, $-1 < \alpha_k < 0$ $(m_1 + 1 \leqslant k \leqslant m)$. The function

$$G_1(t) = \prod_{k=1}^{m} (t - z_0)^{-\lambda_k} G(t), \qquad z_0 \in D^+,$$

where the factor with number k is regarded as discontinuous at the point t_k, will be a nonzero function belonging to the space H_α.

Let us rewrite the condition of problem (1) in the form

$$f^+(t) = \prod_{k=1}^{m} (t - z_0)^{\lambda_k} G_1(t) f^-(t) + g(t)$$

and, using the known representation

$$\prod_{k=1}^{m} (t - z_0)^{\lambda_k} = \frac{\omega^+(t)}{\omega^-(t)},$$

where

$$\omega^+(z) = \prod_{k=1}^{m} (z - t_0)^{\lambda_k}, \qquad \omega^-(z) = \prod_{k=1}^{m} \left(\frac{z - t_k}{z - z_0} \right)^{\lambda_k},$$

we bring the boundary condition of the original problem to the form

$$F^+(t) = G_1(t) F^-(t) + g(t)/\omega^+(t). \tag{8}$$

In the boundary condition (8), $F^\pm(t)$ are the boundary values of the analytic functions $F^+(z) = f^+(z)/\omega^+(z)$, $F^-(z) = f^-(z)/\omega^-(z)$; the coefficient $G_1(t) \in H_a$, and the free term $g(t)/\omega^+(t)$ is a function summable to the power p $(p > 1)$ with weight

$$\prod_{k=1}^{m_1} |t - t_k|^{\beta_k (p-1)} \prod_{k=m_1+1}^{m} |t - t_k|^{-\beta_k},$$

if the function $g(t)$ was summable to the power p with weight

$$\prod_{k=1}^{m_1} |t-t_k|^{-\beta_k} \prod_{k=m_1+1}^{m} |t-t_k|^{\beta_k(p-1)},$$

where $\beta_k = \alpha_k$ for $k \leq m$, and $\beta_k = -\alpha_k$ for $m_1 + 1 \leq k \leq m$.

We shall prove that, in the case under consideration, problem (1) will have no solutions in H'_α other than the classical ones. For the proof, note that between the solutions of problems (1) and (8) there is a one-to-one correspondence. Consequently, it suffices for us to prove that all solutions of problem (8) in the class H'_α are exhausted by the functionals corresponding to classical solutions, and this was proved at the end of the preceding paragraph.

In conclusion, let us note that nowhere have we used the fact that generalized functions from the subspaces H'^{+}_α, H'^{-}_α are, in any sense, boundary values of functions analytic respectively in the domains D^+, D^-. The study of such a connection between functionals and analytic functions is of independent interest. For functionals defined on the space of infinitely differentiable functions, such a connection has been well studied ([5,6]).

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Note: Figure translations are in progress. See original paper for figures.