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CYBERNETICS AND CONTROL THEORY

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Abstract

Full Text

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CYBERNETICS AND CONTROL THEORY

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ON THE THEORY OF NECESSARY CONDITIONS FOR THE OPTIMALITY OF SINGULAR CONTROLS

(Presented by Academician L. S. Pontryagin on 13 III 1968)

1. Statement of the problem. Let the equations of motion be given

$$\dot{x}_i = f_i(x, u, t), \qquad x_i(t_0) = x_{i0},$$

where $x = \{x_1, \dots, x_n\}$, $u = \{u_1, \dots, u_r\}$; $f_i(x, u, t)$ are defined and continuous on $X \times U \times T$, together with $\partial f_i(x, u, t)/\partial x_j$; X, U, T are the domains of variation of the quantities x, u, t.

It is required, among the piecewise-continuous r-dimensional vector functions $u(t)$, defined on $T = [t_0, t_1]$ with values in the bounded set U, to find such a $u^0(t)$, $t \in T$, that gives the functional

$$I(u) = \varphi(x(t_1))$$

($\varphi(x)$ differentiable on X) its minimal value.

It is known ([1]) that the optimal control $u^0(t)$ and the corresponding optimal trajectory $x^0(t)$ satisfy the maximum principle of L. S. Pontryagin

$$H(x^0(t), \psi^0(t), u^0(t), t) = \sup_{u \in U} H(x^0(t), \psi^0(t), u, t),$$

$$H(x, \psi, u, t) = \psi_i f_i(x, u, t),$$

$$\dot{\psi}_i^0(t) = -\frac{\partial f_j(x^0(t), u^0(t), t)}{\partial x_i} \psi_j^0(t), \qquad \psi_i(t_1) = -\frac{\partial \varphi(x^0(t_1))}{\partial x_i}.$$

Let the control $u^*(t)$, $t \in T$, be such that the function $H(x^*(t), \psi^*(t), u, t)$ for $t \in T$ does not depend on u:

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$$H(x^*(t), \psi^*(t), u, t) - H(x^*(t), \psi^*(t), u^*(t), t) \equiv 0, \qquad t \in T, \qquad u \in U.$$

Such a control $u^*(t)$ will be called a **singular control of the first order**.

2. Necessary optimality conditions for a singular control of the first order. Following the scheme of ([2,3]), we construct a formula for the increment of the functional, use a needle variation of the control concentrated on an interval of length ε, and isolate in the increment of the functional the terms of order ε^2. As a result we arrive at the following assertion.

Theorem 1. Let the functions $f_i(x, u, t)$, $\varphi(x)$ be defined and continuous on $X \times U \times T$, together with $\partial f_i(x, u, t)/\partial x_j$, $\partial \varphi(x)/\partial x_i$, $\partial^2 f_i(x, u, t)/\partial x_j \partial x_k$, $\partial^2 \varphi(x)/\partial x_i \partial x_j$. Then, for a singular optimal control of the first order $u^0(t)$, $t \in T$, the conditions

$$\psi_j(t) \frac{\partial \Delta_\nu f_j(x^0(t), u^0(t), t)}{\partial x_i} \Delta_x f_i(x^0(t), u^0(t), t) +$$

$$+ [M_{ji}(t) + M_{ij}(t)] \Delta_x f_i(x^0(t), u^0(t), t) \Delta_x f_j(x^0(t), u^0(t), t) \leq 0, \qquad (1)$$

$$t \in T, \qquad \nu \in U;$$

$$\dot{\psi}_i(t) = -\frac{\partial f_j(x^0(t), u^0(t), t)}{\partial x_i} \psi_j(t), \qquad \psi_i(t_1) = -\frac{\partial \varphi(x^0(t_1))}{\partial x_i};$$

$$\dot{M}_{ij}(t) = -M_{ik}(t) \frac{\partial f_k(x^0(t), u^0(t), t)}{\partial x_j} - M_{kj}(t) \frac{\partial f_k(x^0(t), u^0(t), t)}{\partial x_i}$$
$$-\frac{1}{2} \psi_k(t) \frac{\partial^2 f_k(x^0(t), u^0(t), t)}{\partial x_i \partial x_j}, \qquad M_{ij}(t_1) = -\frac{1}{2} \frac{\partial^2 \varphi(x^0(t_1))}{\partial x_i \partial x_j};$$
$$\Delta_v f_i(x, u, t) = f_i(x, v, t) - f_i(x, u, t).$$

3. Singular controls of second order. A control $u^*(t)$, $t \in T$, is called a singular control of second order if identically with respect to $t \in T$, $v \in U$,

$$\psi_j(t) \Delta_v f_j(x^*(t), u^*(t), t) \equiv 0,$$

$$\psi_j(t) \frac{\partial \Delta_v f_j(x^*(t), u^*(t), t)}{\partial x_i} + [M_{ij}(t) + M_{ji}(t)] \Delta_v f_j(x^*(t), u^*(t), t) \equiv 0.$$

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To obtain necessary conditions for optimality of singular controls of second order in the increment of the functional caused by a needle variation of the control, we isolate the terms of order ε^3.

Theorem 2. Let $f_i(x,u,t)$, $\varphi(x)$ be defined and continuous on $X \times U \times T$, together with $\partial f_i(x,u,t)/\partial x_j$, $\partial \varphi(x)/\partial x_i$, $\partial^2 f_i(x,u,t)/\partial x_j \partial x_k$, $\partial^2 \varphi(x)/\partial x_i \partial x_j$, $\partial^3 f_i(x,u,t)/\partial x_j \partial x_k \partial x_l$, $\partial^3 \varphi(x)/\partial x_i \partial x_j \partial x_k$.

If $u^0(t)$, $t \in T$, is a singular optimal control of second order, then for all $t \in T$, $v \in U$ the conditions

$$\begin{aligned}&\frac{1}{2}\psi_k(t)\frac{\partial^2 \Delta_v f_k^0(t)}{\partial x_i \partial x_j}\Delta_v f_i^0(t)\Delta_v f_j^0(t)+\\ &\quad+[M_{ik}(t)\partial \Delta_v f_k^0(t)/\partial x_j + M_{kj}(t)\partial\Delta_v f_k^0(t)/\partial x_i]\Delta_v f_i^0(t)\Delta_v f_j^0(t)+\\ &\quad+[N_{ijk}(t)+N_{ikj}(t)+N_{kji}(t)]\Delta_v f_i^0(t)\Delta_v f_j^0(t)\Delta_v f_k^0(t)\leqslant 0;\end{aligned} \tag{2}$$

$$\dot{\psi}_i(t) = -\frac{\partial f_j^0(t)}{\partial x_i}\psi_j(t), \qquad \psi_i(t_1) = -\frac{\partial \varphi(x^0(t_1))}{\partial x_i};$$

$$\dot{M}_{ij}(t) = -M_{ik}(t)\frac{\partial f_k^0(t)}{\partial x_j} - M_{kj}(t)\frac{\partial f_k^0(t)}{\partial x_i} - \frac{1}{2}\psi_k(t)\frac{\partial^2 f_k^0(t)}{\partial x_i \partial x_j};$$

$$M_{ij}(t_1) = -\frac{1}{2}\frac{\partial^2 \varphi(x^0(t_1))}{\partial x_i \partial x_j};$$

$$\begin{aligned}\dot{N}_{ijk}(t) = &- N_{ijl}(t)\frac{\partial f_l^0(t)}{\partial x_k} - N_{ilk}(t)\frac{\partial f_l^0(t)}{\partial x_j} - N_{ljk}(t)\frac{\partial f_l^0(t)}{\partial x_i}\\ &- \frac{1}{2}M_{il}(t)\frac{\partial^2 f_l^0(t)}{\partial x_k \partial x_j} - \frac{1}{2}M_{lj}(t)\frac{\partial^2 f_l^0(t)}{\partial x_k \partial x_i} - \frac{1}{6}\psi_l(t)\frac{\partial^3 f_l^0(t)}{\partial x_i \partial x_j \partial x_k};\end{aligned}$$

$$N_{ijk}(t_1) = -\frac{1}{6}\frac{\partial^3 \varphi(x^0(t_1))}{\partial x_i \partial x_j \partial x_k}.$$

Here $f_i^0(t) = f_i(x^0(t), u^0(t), t)$.

4. The case of a convex control domain. Let the set U be convex, and let the functions $f_i(x,u,t)$, $\partial f_i(x,u,t)/\partial x_j$ be differentiable with respect to u. Then in Theorem 1 assertion (1) may be replaced by the following:

$$p_{kl}(t)[v_k - u_k^0(t)][v_l - u_l^0(t)] \leqslant 0, \qquad t \in T, \quad v \in U;$$

$$p_{kl}(t) = \psi_j(t)\frac{\partial^2 f_j^0(t)}{\partial x_i \partial u_k}\frac{\partial f_i^0(t)}{\partial u_l} + [M_{li}(t) + M_{ji}(t)]\frac{\partial f_j^0(t)}{\partial u_k}\frac{\partial f_i^0(t)}{\partial u_l}. \tag{3}$$

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If, in addition, $\partial^2 f_i(x,u,t)/\partial x_j \partial x_k$ are differentiable with respect to u, then inequality (2) of Theorem 2 takes the form

$$q_{lms}(t)[v_l - u_l^0(t)][v_m - u_m^0(t)][v_s - u_s^0(t)] \leqslant 0, \qquad t \in T, \quad v \in U; \qquad (4)$$

$$\begin{aligned} q_{lms}(t) = &\frac{1}{2}\psi_k(t)\frac{\partial^3 f_k^0(t)}{\partial x_i \partial x_j \partial u_l}\frac{\partial f_i^0(t)}{\partial u_m}\frac{\partial f_j^0(t)}{\partial u_s} \\ &+ \left[M_{ik}(t)\frac{\partial^2 f_k^0(t)}{\partial x_j \partial u_l} + M_{kj}(t)\frac{\partial^2 f_k^0(t)}{\partial x_i \partial u_l}\right]\frac{\partial f_i^0(t)}{\partial u_m}\frac{\partial f_j^0(t)}{\partial u_s} \\ &+ [N_{ijk}(t) + N_{ikj}(t) + N_{kji}(t)]\frac{\partial f_i^0(t)}{\partial u_l}\frac{\partial f_j^0(t)}{\partial u_m}\frac{\partial f_k^0(t)}{\partial u_s}. \end{aligned}$$

5. **The case of an open control domain.** Let $u^0(t)$ be an interior point of the set U. Then, from property (3) of the optimal control, it follows that the quadratic form $p_{kl}(t)u_k u_l$, $t \in T$, is nonpositive; from (4), that the form $q_{klm}(t)u_k u_l u_m$, $t \in T$, is nonpositive.

Singular controls for an open set U were studied in ([4–7]). The necessary optimality conditions obtained above differ from the known ones both in form and in the method of their derivation. The variations of controls used in ([4–7]) essentially presuppose that U is an open set.

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