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Abstract

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Parametric Resonance in Canonical Systems of Linear Differential Equations with Quasiperiodic Coefficients

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Consider the canonical system

$$\frac{dz}{dt} = JH(\varphi, \mu)z, \qquad \varphi = \omega t. \tag{1}$$

Here $z = (z_1, \dots, z_{2n})$, $\varphi = (\varphi_1, \dots, \varphi_m)$, $\omega = (\omega_1, \dots, \omega_m)$ are vectors; $H(\varphi, \mu)$ is a real symmetric matrix-function, periodic in $\varphi_1, \dots, \varphi_m$;

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

I_n is the identity matrix of order n; $\mu \geq 0$ is a small parameter. Without loss of generality one may assume the frequencies $\omega_1, \dots, \omega_m$ to be linearly independent over the set of integers. Denote by W_q the set of matrix-functions $F(\varphi, \mu)$ admitting the representation

$$F(\varphi, \mu) = \sum_{k=0}^{\infty} \mu^k \sum_{0 \leq |p| \leq N_k} F^{(p)} e^{i(p, \varphi)}, \tag{2}$$

$$\left(|p| = \sum_{s=1}^{m} |p_s|, \qquad (p, \varphi) = \sum_{s=1}^{m} p_s \varphi_s \right)$$

(p is a vector with integer components) and such that, for $\mu q \leq 1$, the finite sum $M(F)$ of the series

$$\sum_{k=0}^{\infty} \mu^k \sum_{0 \leq |p| \leq N_k} |F^{(p)}|,$$

where $|F^{(p)}|$ is the norm of the matrix $F^{(p)}$ (for definiteness we shall use the third matrix norm (^{1})). It is assumed that $H(\varphi, \mu)$ in (1) has the structure

$$H(\varphi, \mu) = \Lambda + F(\varphi, \mu), \qquad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \tag{3}$$

with $F(\varphi, \mu) \in W_q$ and $M(F) < \mu q$.

In the "nonresonant case" (i.e., when $\lambda_i + \lambda_j + (p, \omega) \neq 0$ for all i, j and $\lambda_i - \lambda_j + (p, \omega) \neq 0$ for $i \neq j$), the usual methods of perturbation theory, going back to Poincaré (^{2}), give for the characteristic exponents of system (1), (3) asymptotic estimates of order μ^k, where k is any natural number. A somewhat more delicate method makes it possible to obtain the same estimates while restricting oneself to the assumption $\lambda_i + \lambda_j + (p, \omega) \neq 0$.

Theorem 1. *Let $\lambda_i + \lambda_j + (p, \omega) \neq 0$ $(1 \leq i, j \leq n)$ for all vectors p with integer components. Then for any natural r there exists a constant $q_r < \infty$ such that, if the inequality $\mu q_r < 1$ is satisfied, every solution $z(t, \mu)$ of system (1), (3) admits the estimate*

$$|z(t, \mu)| \leq |z(0, \mu)|(1 + \mu q_r) e^{|t|(\mu q_r)^r} \left(|z| = \left(\sum_{s=1}^{2n} |z_s|^2 \right)^{1/2} \right). \tag{4}$$

Remark 1. If the "perturbed" system (1), (3) is not canonical (i.e., if the matrix F in (3) is nonsymmetric), then the estimate (4), generally speaking, is false when there is at least one relation

$$\lambda_i - \lambda_j + (p, \omega) = 0.$$

Remark 2. Let $m = 1$ (a system with periodic coefficients). Then it follows from the proof of the theorem that in this case $q_r < d < \infty$ for all r. Passing in (4) to the limit as $r \to \infty$, $\mu d < 1$, we conclude that the system (1), (3) is stable (has only bounded solutions). This result constitutes the content of a well-known theorem of M. G. Krein (^{3}).

We turn to a brief exposition of the proof of the theorem. To each matrix $F(\omega t)$ we associate the matrix $[F]$ according to the rule

$$[F] = e^{-L' t} B e^{Lt}, \qquad B = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{L' t} F(\omega t) e^{Lt} \, dt, \qquad L = J\Lambda.$$

(Here the prime denotes the transposition operation.) It can be shown that $[F]$ is a quasiperiodic matrix function with frequency basis $\omega_1, \dots, \omega_m$. Put $\widetilde{F} = F - [F]$. Obviously, the identity $[\widetilde{F}] = 0$ holds. The proof of Theorem 1 is based on two lemmas, which are of independent interest.

Lemma 1. For any natural number $r \geq 1$ there exists a constant $q_r < \infty$ such that, if the inequality $\mu q_r < 1$ is satisfied, then there exists a symplectic substitution

$$z = Q^{(r)}(\omega t, \mu) z^{(r)}, \qquad Q^{(r)} \in W_{q_r},$$

which brings (1), (3) to the form

$$dz^{(r)}/dt = JH^{(r)} z^{(r)}, \qquad H^{(r)} = \Lambda + F^{(r)}(\varphi, \mu), \qquad \varphi = \omega t, \qquad F^{(r)} \in W_{q_r}, \tag{5}$$

and the estimates

$$M(\widetilde{F}^{(r)}) < (\mu q_r)^r, \qquad M(I_{2n} - Q^{(r)}) < \mu q_r$$

are valid.

The proof is by induction. For $r = 1$ the assertion of the lemma is obviously fulfilled if one puts $Q^{(1)} = I_{2n}$, $H^{(1)} = H$. We seek the symplectic substitution $z^{(r)} \to z^{(r+1)}$, possessing the required properties, in the form

$$z^{(r)} = e^{JS} z^{(r+1)}$$

($S(\omega t)$ is an unknown symmetric matrix). In doing so, one has to solve the equation

$$L'S + SL + dS/dt = \widetilde{R}, \tag{6}$$

where $R(\omega t)$ is a trigonometric polynomial with matrix coefficients. It can be shown that, owing to the identity $[\widetilde{R}] = 0$, equation (6) is always solvable and determines $S(\omega t)$ in the form of a trigonometric polynomial.

Lemma 2. There exists a matrix

$$\Gamma = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \qquad \gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n),$$

possessing the following properties:

1. $$e^{J\Gamma t} = \begin{pmatrix} \cos \gamma t & \sin \gamma t \\ -\sin \gamma t & \cos \gamma t \end{pmatrix}$$

   is a quasiperiodic matrix function with frequency basis

   $$\tfrac{1}{2}\omega_1, \ldots, \tfrac{1}{2}\omega_m.$$

2. The substitution $z^{(r)} = e^{J\Gamma t}\xi$ brings system (5) to the form

   $$d\xi/dt = J[A^{(r)}(\mu) + G^{(r)}(\varphi, \mu)]\xi, \qquad \varphi = \omega t/2, \tag{7}$$

   where $A^{(r)}(\mu)$ is a constant matrix and

   $$|G^{(r)}| \leq M(\widetilde{F}^{(r)}) < (\mu q_r)^r.$$

3. If

$$\lambda_i + \lambda_j + (p, \omega) \neq 0 \qquad (1 \leq i, j \leq n),$$

then $e^{J\Gamma t}$ has frequency basis $\omega_1, \dots, \omega_m$. In this case the matrix $JA^{(r)}$ is skew-symmetric.

To construct the matrix Γ, it suffices to find constants $\gamma_1, \dots, \gamma_n$ satisfying the same resonance relations as $\lambda_1, \dots, \lambda_n$, and which are linear combinations of $\frac{1}{2}\omega_1, \dots, \frac{1}{2}\omega_m$ with integer coefficients. Such constants can be found by arranging in a definite way the set of resonance relations.

Theorem 1 is a simple consequence of Lemmas 1 and 2, since, if $\lambda_i + \lambda_j + (p, \omega) \neq 0$, then the matricant $Z(t, \mu)$ of the system (1), (3) admits the estimate

$$|Z(t,\mu)| \leq |Q^{(r)}(\omega t, \mu)|\,|e^{J\Gamma t}|\,|e^{JA^{(r)}t}|\,e^{Mt} < (1 + \mu q_r)e^{Mt},$$

where

$$M = \max_t |e^{-JA^{(r)}t} JG^{(r)} e^{JA^{(r)}t}| < \max_t |G^{(r)}| < (\mu q_r)^r$$

(we use the identity $|e^{J\Gamma t}| = |e^{JA^{(r)}t}| = 1$, which follows from the orthogonality of the matrices $e^{J\Gamma t}$, $e^{JA^{(r)}t}$).

Theorem 1 can be supplemented by the following two assertions:

Theorem 2. *If at least one relation*

$$\lambda_i + \lambda_j + (p, \omega) = 0$$

is satisfied, then there exists a symmetric matrix $F_1(\varphi, \mu) \in W_q$ such that the system (1), (3) is unstable for $F = F_1$, $\mu q \leq 1$. If $M(F_2 - F_1) < \varepsilon$ and ε is sufficiently small, then the system (1), (3) is unstable for $F = F_2$.

Theorem 3. *Let there be a unique resonant relation*

$$2\lambda_1 + (p, \omega) = 0$$

("simple resonance"). Then, if for some $r \geq 1$ the inequality

$$\det(JA^{(r)}) < 0$$

holds ($A^{(r)}$ is the matrix appearing in (7)), then, for sufficiently small μ, the system (1), (3) has two particular real solutions

$$z = f_1\left(\tfrac{1}{2}\omega t\right) e^{\nu(t)}, \qquad z = f_2\left(\tfrac{1}{2}\omega t\right) e^{-\nu t},$$

where $f_1(\varphi),\ f_2(\varphi)$ *are* 2π*-periodic vector-functions;*

$$\nu(t) = \nu_0 t + \int_0^t f\left(\tfrac{1}{2}\omega t\right)\,dt;$$

$f(\varphi)$ *is a* 2π*-periodic function with zero mean value, and the estimates*

$$|\nu_0| > c_1\mu^r, \qquad |f| < c_2\nu_0\mu$$

hold ($c_1 > 0,\ c_2$ *are constants independent of* μ).

The proof of these assertions is based on Lemma 3, concerning a system of the form

$$\frac{dz}{dt} = JH(\varphi)z, \qquad \varphi = \tilde{\omega}t, \qquad H = \Lambda + F(\varphi),$$

$$\Lambda = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n). \tag{8}$$

The matrix $H(\varphi)$ in (8) is, generally speaking, complex-valued. We shall assume that at least one of the numbers $\operatorname{Re}\sigma_1, \dots, \operatorname{Re}\sigma_n$ is different from zero, and require that the elements of the matrix-function $F(\varphi)$ be expandable in absolutely convergent Fourier series. Put

$$\|F\| = \sum_{0\le|p|} |F^{(p)}|,$$

where

$$F^{(p)} = \frac{1}{(2\pi)^m}\int_0^{2\pi} F(\varphi)e^{i(p,\varphi)}\,d\varphi$$

is the matrix of Fourier coefficients. To each matrix $F(\varphi)$ we associate the matrix $\{F\}$

$$\{F\} = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\,\operatorname{Re}LT}F(\varphi)e^{i\,\operatorname{Re}Lt}\,dt, \qquad L = J\Lambda = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}.$$

Put $\hat{F} = F - \{F\}$. Let

$$\rho = \min_{i,j}|\operatorname{Re}(\sigma_i \pm \sigma_j)|,$$

where the minimum is taken only over those i, j for which $\mathrm{Re}(\sigma_i \pm \sigma_j) \neq 0$.

Lemma 3. *If* $c\|\hat{F}\| < \rho^3$ *(*c *is some constant independent of* ρ*), then there exists a substitution* $z = P(\varphi)\zeta$, $\varphi = \omega t$, *which reduces the system (8) to the form*

$$\frac{d\zeta}{dt} = JH_1(\varphi)\zeta, \qquad \varphi = \omega t, \qquad H_1 = \Lambda + \{G\}, \tag{9}$$

with $\|G\| < c\|F\|$.

The proof of Lemma 3 is, from the formal point of view, analogous to the proof of Lemma 1. However, in the present case, instead of equation (6) one has to solve the equation $LS + SL + dS/dt = \hat{F}$. In solving this equation no "small divisors" appear, which is due to the special structure of the matrix $\hat{F}$; as a result of this circumstance it is possible to construct a convergent process. The significance of Lemma 3 is that system (9) decomposes into k independent subsystems, where k is the number of groups of the numbers $\sigma_1, \dots, \sigma_n$ connected by the relation $\mathrm{Re}(\sigma_i \pm \sigma_j) = 0$. In particular, if $\sigma_1 \neq 0$ is a real number and $\sigma_1 \pm \mathrm{Re}\,\sigma_i \neq 0$ $(i \neq 1)$, then the equations with respect to ζ_1 and ζ_{n+1} do not depend on the remaining variables and are easily integrated. Combining this result with the assertions of Lemmas 1 and 2, we obtain Theorem 3. The condition $\det(JA^{(r)}) < 0$ of this theorem guarantees the existence of one pair of real eigenvalues of the matrix $JA^{(r)}$ (under the condition that there is a single resonance relation $2\lambda_1 + (p, \omega) = 0$).

Theorem 2 is proved by constructing a concrete system (1), (3), which, in the presence of the resonance relation $\lambda_i + \lambda_j + (p, \omega) = 0$, is reduced to the form (8), and then, with the aid of Lemma 3, to the form (9). The instability of system (9) is proved quite simply if one takes into account the special structure of the matrix $\hat{F}$.

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