BOUNDARY-VALUE PROBLEMS OF LINEAR CONJUGATION FOR FUNCTIONS HOLOMORPHIC IN BICYLINDRICAL DOMAINS

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Abstract

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MATHEMATICS

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BOUNDARY-VALUE PROBLEMS OF LINEAR CONJUGATION FOR FUNCTIONS HOLOMORPHIC IN BICYLINDRICAL DOMAINS

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1°. Put

$$F(z,w) = \frac{1}{(2\pi i)^2}\int_{C\times\Gamma}\frac{f(t,\omega)\,dt\,d\omega}{(t-z)(\omega-w)} \equiv K(f), \qquad z\in C, \quad w\in\Gamma,$$

$$S_t f = \frac{1}{\pi i}\int_C \frac{f(t_1,\omega)}{t_1-t}\,dt_1, \quad t\in C; \qquad S_\omega f = \frac{1}{\pi i}\int_\Gamma \frac{f(t,\omega_1)}{\omega_1-\omega}\,d\omega_1, \quad \omega\in\Gamma,$$

$S = S_t S_\omega = S_\omega S_t$, where $C = \partial D^+$ ($\Gamma = \partial\Delta^+$) is a simple smooth closed contour; $C\times\Gamma$ is the skeleton of the boundaries of the bicylindrical domains $D^+\times\Delta^+$, $D^+\times\Delta^-$, $D^-\times\Delta^+$, $D^-\times\Delta^-$, oriented in the natural way; D^- (Δ^-) complements $D^+\cup C$ ($\Delta^+\cup\Gamma$) to the full complex plane of the variable z (w), $z=0\in D^+$ ($w=0\in\Delta^+$); $f(t,\omega)$ is a function satisfying the Hölder condition on $C\times\Gamma$ (briefly, $f\in H$). By $H^{\pm\pm}$ ($H^{\pm\mp}$) we denote the class of functions holomorphic in $D^\pm\times\Delta^\pm$ ($D^\pm\times\Delta^\mp$), whose limiting values on $C\times\Gamma$ belong to the class H; $H_0^{\pm\mp}$ (H_0) is the subclass of functions from $H^{\pm\mp}$ (H^{--}) vanishing at infinitely distant points. The classes $H(C)$, $H^\pm(C)$, $H_0^-(C)$ ($H(\Gamma), H^\pm(\Gamma), H_0^+(\Gamma)$) of functions depending on one variable z (w) have an analogous meaning. Functions of the class $H^{\pm\pm}$ ($H^{\pm\mp}$) will be denoted by $F^{\pm\pm}(z,w)$ ($F^{\pm\mp}(z,w)$).

Put further

$$F^{\pm\pm}(t,\omega) = \lim_{(z,w)\to(t,\omega)} F^{\pm\pm}(z,w), \qquad (z,w)\in D^\pm\times\Delta^\pm,$$

$$F^{\pm\mp}(t,\omega) = \lim_{(z,w)\to(t,\omega)} F^{\pm\mp}(z,w), \qquad (z,w)\in D^\pm\times\Delta^\mp,$$

where $(t,\omega)\in C\times\Gamma$,

$$D_0^\pm = \{(z,w): z\in D^\pm,\ w=0\}, \qquad D_\infty^\pm = \{(z,w): z\in D^\pm,\ w=\infty\},$$

$$\Delta_0^{\pm} = \{(z,w) : w \in \Delta^{\pm},\ z = 0\}, \qquad \Delta_\infty^{\pm} = \{(z,w) : w \in \Delta^{\pm},\ z = \infty\},$$

$$\frac{\partial}{\partial \bar{z}} = -z^2 \frac{\partial}{\partial z}, \qquad \frac{\partial^k}{\partial \bar{z}^k} = \frac{\partial^{k-1}}{\partial \bar{z}^{k-1}} \left(\frac{\partial}{\partial \bar{z}} \right), \qquad k = 2, 3, \dots$$

2°. The problem of finding functions $\Phi^{\pm\pm}(z,w)$ and $\Phi^{\pm\mp}(z,w)$ from the linear relation

$$A(t,\omega)\Phi^{++}(t,\omega)+B(t,\omega)\Phi^{-+}(t,\omega)+C(t,\omega)\Phi^{+-}(t,\omega)+D(t,\omega)\Phi^{--}(t,\omega) = f(t,\omega), \quad (t,\omega) \in C\times\Gamma, \tag{1}$$

for $f \not\equiv 0$ ($f \equiv 0$) will be called a nonhomogeneous (homogeneous) problem of linear conjugation for bicylindrical domains.

Consider the principal homogeneous elementary problem: $A = 1$, $B = -t^r$, $C = -\omega^\nu$, $D = t^m \omega^\mu$, $m \geq r > 0$, $\mu \geq \nu > 0$, m, r, μ, ν are integers. The general solution of this problem in the classes $H_0^{\pm\mp}$, H_0^{--}, and H^{++} is given by the formulas:

$$\begin{aligned}
\Phi^{++}(z,w) &= \sum_{k=0}^{r-1} z^k \varphi_k^+(w) + \sum_{j=0}^{\nu-1} w^j \psi_j^+(z) + \sum_{k=0}^{m-1}\sum_{j=0}^{\mu-1} c_{kj} z^k w^j,\\
z^r \Phi^{-+}(z,w) &= \sum_{k=0}^{r-1} z^k \varphi_k^+(w) + \sum_{j=0}^{\mu-1} w^j a_j^-(z),\\
w^\nu \Phi^{+-}(z,w) &= \sum_{k=0}^{\nu-1} w^k \psi_k^+(z) + \sum_{j=0}^{m-1} z^j b_j^-(w),\\
z^m w^\mu \Phi^{--}(z,w) &= \sum_{k=0}^{m-1} z^k b_k^-(w) + \sum_{j=0}^{\mu-1} w^j a_j^-(z) - \sum_{k=0}^{m-1}\sum_{j=0}^{\mu-1} c_{kj} z^k w^j,
\end{aligned} \tag{2}$$

where $\varphi_k^+(w)$, $\varphi_k^+(z)$ ($b_k^-(w)$, $a_k^-(z)$) are arbitrary functions of class $H^+(\Gamma)$, $H^+(C)$ ($H_0^-(\Gamma)$, $H_0^-(C)$), and c_{kj} are arbitrary constants.

The problem thus posed is underdetermined. If to condition (1) one adjoins boundary conditions of the type of Cauchy conditions for partial differential equations, namely:

$$\begin{aligned}
&\left.\frac{\partial^k \Phi^{-+}(z,w)}{\partial z^k}\right|_{D_\infty^+} = k!\varphi_{r-k}^+(w), \quad k = 1, \dots, r, \qquad \left.\frac{\partial^j \Phi^{+-}(z,w)}{\partial w^j}\right|_{D_\infty^+} = j!\psi_{\nu-j}^+(z), \quad j = 1, \dots, \nu,\\
&\left.\frac{\partial^l \Phi^{--}(z,w)}{\partial z^l}\right|_{D_\infty^-} = l!b_{m-l}^-(w), \quad l = 1, \dots, m, \qquad \left.\frac{\partial^i \Phi^{--}(z,w)}{\partial w^i}\right|_{D_\infty^-} = i!a_{\mu-i}^-(z), \quad i = 1, \dots, \mu,
\end{aligned} \tag{3}$$

where φ_k^+, ψ_j^+, b_l^-, a_i^- are prescribed functions of the corresponding class, then the basic elementary homogeneous problem has $m\mu$ linearly independent solutions (2), vanishing at infinitely remote points.

Theorem 1. *The basic homogeneous elementary problem has a countable number of linearly independent solutions. It has a finite number of such solutions if one seeks solutions satisfying boundary conditions of the form (3).*

3°. The problem $A = 1$, $B = -t^r\omega^\rho$, $C = -t^n\omega^\nu$, $D = t^m\omega^\mu$ and $f \equiv 0$ will be called the **homogeneous elementary** problem. Such a problem can always be reduced to the basic elementary one. Let, for example, $r \leq 0$, $m > 0$, $\nu > 0$, $\mu < \nu$; then, putting $\Phi^{++} = \Psi^{++}$, $t^r\omega^\rho\Phi^{-+} = \Psi^{-+}$, $t^n\Phi^{+-} = \Psi^{+-}$, $\omega^{\mu-\nu}\Phi^{--} = \Psi^{--}$, we obtain the basic elementary problem with $r = 0$, $m > 0$ and $\nu = \mu > 0$, whose solution $\Psi^{\pm\pm}$ and $\Psi^{\pm\mp}$ is found by formulas (2), omitting in them the sums $\sum z^k\varphi_k^+$ and then requiring that

$$\Phi^{-+} = z^{-r}w^{-\rho}\Psi^{-+} \in H_0^+, \qquad \Phi^{+-} = z^{-n}\Psi^{+-} \in H_0^+, \qquad \Phi^{--} = w^{\nu-\mu}\Psi^{--} \in H_0^-,$$

which leads to restrictions on the functions ψ_j^+, b_l^-, a_i^-.

The nonhomogeneous elementary problem, after the substitution

$$\Psi^{++} = \Phi^{++} - F^{++}, \quad \Psi^{-+} = \omega^\rho\Phi^{-+} - t^{-r}F^{-+}, \quad \Psi^{+-} = t^n\Phi^{+-} - \omega^{-\nu}F^{+-}, \quad \Psi^{--} = \Phi^{--} - t^{-m}\omega^{-\mu}F^{--},$$

where $f = F^{++} - F^{-+} - F^{+-} + F^{++}$, and, for simplicity, $m \geq r > 0$, $\mu \geq \nu > 0$, is reduced to the basic homogeneous elementary problem. The formulas giving the solution of this problem satisfying boundary conditions of the form (3) are cumbersome and are not given here.

Theorem 2. *The nonhomogeneous elementary problem is solvable provided no more than a countable number of necessary and sufficient solvability conditions are fulfilled, conditions which the free term $f(t,\omega)$ must satisfy.*

When these conditions are fulfilled, the problem has no more than a countable number of linearly independent solutions. The number of solutions satisfying conditions of the form (3) is finite.

4°. Let the function $G(t,\omega) \neq 0$ be continuous on $C \times \Gamma$; then any branch of the function $\ln G(t,\omega)$ is single-valued if the partial indices are

$$l(G) = \frac{1}{2\pi i}\int_C d\ln G(t,\omega), \qquad \lambda(G) = \frac{1}{2\pi i}\int_\Gamma d\ln G(t,\omega),$$

which are integers, are equal to zero.

5°. Let, in the condition of linear conjugation (1), $f = 0$ and

$$
\begin{array}{lll}
\text{(A)} & B = C = 0, & D/A = -G_1(t,\omega) \neq 0. \\
\text{( )} & A = D = 0, & B/C = -G_2(t,\omega) \neq 0. \\
\text{( )} & C = D = 0, & B/A = -G_3(t,\omega) \neq 0. \\
\text{( )} & B = D = 0, & C/A = G_4(t,\omega) \neq 0. \\
\text{( )} & A = C = 0, & D/B = -G_5(t,\omega) \neq 0. \\
\text{( )} & A = B = 0, & D/C = -G_6(t,\omega) \neq 0.
\end{array}
$$

Such problems will be called **degenerate homogeneous problems**.

Put $l_k = l(G_k)$, $\lambda_k = \lambda(G_k)$, $k = 1, 2, 3, 4, 5, 6$.

Theorem 3. Let $l_k = \lambda_k = 0$, $k = 1, 2, 3, 4, 5, 6$, and, respectively,

$$
\begin{array}{ll}
\text{(A)} & \ln G_1 = S(\ln G_1). \\
\text{( )} & \ln G_2 = -S(\ln G_2). \\
\text{( )} & \ln G_3 = S_\omega(\ln G_3). \\
\text{( )} & \ln G_4 = S_t(\ln G_4). \\
\text{( )} & \ln G_5 = -S_t(\ln G_5). \\
\text{( )} & \ln G_6 = -S_\omega(\ln G_6).
\end{array}
$$

Then the function $G_k \in H$ can be represented as a quotient of two functions

$$G_1 = \Phi_1^{++}/\Phi_1^{--}, \quad G_2 = \Phi_2^{-+}/\Phi_2^{+-}, \quad G_3 = \Phi_3^{++}/\Phi_3^{-+},$$

$$G_4 = \Phi_4^{++}/\Phi_4^{+-}, \quad G_5 = \Phi_5^{-+}/\Phi_5^{--}, \quad G_6 = \Phi_6^{+-}/\Phi_6^{--},$$

where $\Phi_k^{\pm\pm}(z,w) \in H^{\pm\pm}$, $\Phi_k^{\mp\pm}(z,w) \in H^{\mp\pm}$, have no zeros in the corresponding domains, and $\Phi_k^{\mp\pm}$ and Φ_k^{--} tend to one at the infinitely distant points of the domains $D^\mp \times \Delta^\pm$ and $D^- \times \Delta^-$.

Theorem 4. Let $l_1 > 0$ and $\lambda_1 > 0$ ($l_2 > 0$ and $\lambda_2 < 0$). Then the degenerate homogeneous problem (A) (( )) in the class H^{++}, H_0^{--} ($H_0^{\mp\pm}$) has $\chi_1 = l_1\lambda_1$ ($\chi_2 = -l_2\lambda_2$) linearly independent solutions, determined by the formulas

$$\Phi_{r\rho}^{++} = e^{\gamma_{10}^{++}(z,w)} z^r w^\rho, \qquad \Phi_{r\rho}^{--} = e^{-\gamma_{10}^{--}(z,w)} z^{r-l_1} w^{\rho-\lambda_1},$$

$$r = 0, 1, \dots, l_1 - 1, \quad \rho = 0, 1, \dots, \lambda_1 - 1,$$

$$\Big(\Phi_{r\rho}^{+-} = e^{\gamma_{20}^{+-}(z,w)} z^r w^{\lambda_2+\rho}, \qquad \Phi_{r\rho}^{-+} = e^{\gamma_{20}^{-+}(z,w)} z^{r-l_2} w^\rho,$$

$$r = 0, 1, \dots, l_2 - 1, \quad \rho = 0, 1, \dots, -\lambda_2 - 1),$$

where

$$\gamma_{k0}(z,w) = K(G_{k0}), \qquad G_{k0} = G_k t^{-l_k} \omega^{-\lambda_k}, \tag{4}$$

provided that the function G_{10} (G_{20}) satisfies the necessary and sufficient solvability condition $\ln G_{10} = S(\ln G_{10})$ ($\ln G_{20} = -S(\ln G_{20})$).

Theorem 5. If $l_3 > 0$, then problem ( ) in the class H^{++}, H_0^{-+}, for $\lambda_3 \geq 0$ ($\lambda_3 < 0$), has the following general solution:

$$F_k^{++} = e^{\gamma_{30}^{++}(z,w)} \sum_{k=0}^{l_3-1} z^k \varphi_k^+(w), \qquad F_k^{-+} = e^{\gamma_{30}^{-+}(z,w)} \sum_{k=0}^{l_3-1} z^{k-l_3} \varphi_k^+(w) w^{-\lambda_3},$$

where $\gamma_{30}(z,w)$ is determined by formulas (4), and $\varphi_k^+(w)$ ($\varphi_k^+(w) w^{-\lambda_3}$) are arbitrary functions of the class $H^+(\Gamma)$, provided that the function G_{30} satisfies the necessary and sufficient solvability condition $\ln G_{30} = S_\omega(\ln G_{30})$.

For the problems ( ), ( ), and ( ) there are theorems analogous to Theorem 5.

6°. The solvability conditions for the inhomogeneous degenerate problems (A)–( ) are given by the following theorems, which we state only for the typical cases (A) and ( ).

Theorem 6. If $l_1 \geq 0$, $\lambda_1 \geq 0$ and the necessary and sufficient solvability conditions $\ln G_{10} = S(\ln G_{10})$ and $f/\chi^{++} = S(f/\chi^{++})$ are satisfied, where $\chi^{++} = e^{\gamma_{10}^{++}(z,w)}$, $\chi^{--} = e^{-\gamma_{10}^{--}(z,w)} z^{-l_1} \omega^{-\lambda_1}$ are the so-called canonical functions, then the inhomogeneous degenerate problem (A) has in the classes H^{++} and H_0^{--} $\varkappa_1 = l_1 \lambda_1$ linearly independent solutions

$$\Phi_{r\rho}^{\pm\pm} = \chi^{\pm\pm}(z,w) \, [\, \pm \Psi^{\pm\pm}(z,w) + z^r w^\rho \,],$$

$$r = 0, 1, \dots, l_1 - 1, \quad \rho = 0, 1, \dots, \lambda_1 - 1.$$

where $\Psi^{++}(t,\omega) + \Psi^{--}(t,\omega) = f/\chi^{++}$.

If, however, $l_1 < 0, \lambda_1 \geq 0$ ($l_1 \geq 0, \lambda_1 < 0$), then the solution of this problem, when the necessary and sufficient additional conditions

$$\int_{C\times\Gamma} \frac{f(t,\omega)}{\chi^{++}(t,\omega)} \frac{t^{r-1} \, dt \, d\omega}{\omega - w} = 0, \quad w \in \Delta^-, \quad r = 1, \dots, -l_1; \tag{5}$$

$$\left(\int_{C\times\Gamma} \frac{f(t,\omega)}{\chi^{++}(t,\omega)} \frac{\omega^{\rho-1} \, dt \, d\omega}{t - z} = 0, \quad z \in D^-, \quad \rho = 1, \dots, -\lambda_1 \right) \tag{6}$$

are fulfilled, is given by the formulas

$$F^{\pm\pm} = \pm\chi^{\pm\pm}(z,w)\Psi^{\pm\pm}(z,w). \qquad (7)$$

For $l_1 < 0$ and $\lambda_1 < 0$, formulas (7) give a solution if conditions (5) and (6) are satisfied simultaneously.

Theorem 7. If $l_3 \geq 0$ and the necessary and sufficient solvability conditions $\ln G_{30} = S_\omega(\ln G_{30})$ and $f/\chi^{++} = \Psi^{++} - \Psi^{-+}$ are satisfied, where $\chi^{++} = e^{\gamma_{30}^{++}(z,w)}$, $\chi^{-+} = e^{\gamma_{30}^{-+}(z,w)} z^{-l_3} w^{-\lambda_3}$ are canonical functions, then the inhomogeneous degenerate problem (B) in the classes H^{++}, H_0^{-+} has the general solution

$$\Phi^{\pm+} = \chi^{\pm+}(z,w)\left[\Psi^{\pm+}(z,w) + \sum_{k=0}^{l_3-1} z^k \varphi_k^+(w)\right],$$

where arbitrary functions $\varphi_k^+(w) \in H^+(\Gamma)$ $(\varphi_k^+(w) w^{-\lambda_3} \in H^+(\Gamma))$ for $\lambda_3 \leq 0$ $(\lambda_3 > 0)$, and $w^{-\lambda_3}\Psi^{-+}(z,w) \in H_0^{-+}$ for $\lambda_3 > 0$.

If $l_3 < 0$, then the general solution has the form

$$\Phi^{\pm+} = \chi^{\pm+}(z,w)\Psi^{\pm+}(z,w),$$

where the functions $\Psi^{\pm+}$, and hence f and χ^{++}, are such that $z^{-l_3}\Psi^{-+}(z,w) \in H_0^{-+}$ for $\lambda_3 \leq 0$, and $z^{-l_3} w^{-\lambda_3}\Psi^{-+}(z,w) \in H_0^{-+}$ for $\lambda_3 < 0$.

7°. From the theorems formulated above it follows that, in contrast to one-dimensional Riemann problems ([1,2]), the boundary-value problem of linear conjugation for bicylindrical domains is not normally solvable in the sense of Noether. It is normally solvable in the sense of Hausdorff ([3]), and, under boundary conditions of the form (3), has a finite number of solutions. The solvability conditions for inhomogeneous degenerate problems are, in a certain sense, a discrete analogue of the solvability conditions for linear conjugation problems for functions holomorphic in tubular domains ([4]). The assertion, made in ([5]), that the basic elementary homogeneous problem for $\mu = \nu > 0$ and $r = m > 0$, without boundary conditions of the form (3), has a finite number of solutions is incorrect.

The results set forth above were obtained using the author' s work ([6]) and were reported by him at the 4th Section of the International Mathematical Congress ([7]) in Moscow (August 1966).

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Note: Figure translations are in progress. See original paper for figures.