ON GREEN’ S FUNCTIONS FOR PROBLEMS WITH THE BESSEL DIFFERENTIAL OPERATOR

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Abstract

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ON GREEN' S FUNCTIONS FOR PROBLEMS WITH THE BESSEL DIFFERENTIAL OPERATOR

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Let E_{n+2} denote the $(n+2)$-dimensional Euclidean space of points $z=(x,y,t)$, where $x=(x_1,\dots,x_n)$, $x_{n+1}=y$, $x_{n+2}=t$. Consider in this space the domain $y>0,\ t>0$. In this domain consider the following boundary-value problem. Find a solution of the equation

$$\mathcal{L}(D_x,B_y,D_t)=\sum_{|k|=2m} a_k D_x^{k'} B_y^{k_{n+1}} D_t^{k_{n+2}} u=0, \tag{1}$$

where $|k|=|k'|+2k_{n+1}+k_{n+2}$,

$$D_x^{k'}=\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{k_n}, \qquad D_t^{k_{n+2}}=\left(\frac{1}{i}\frac{\partial}{\partial t}\right)^{k_{n+2}},$$

$$B_y^{k_{n+2}}=\left[\left(\frac{1}{i}\right)^2\left(\frac{\partial^2}{\partial y^2}+\frac{2\nu+1}{y}\frac{\partial}{\partial y}\right)\right]^{k_{n+1}}$$

in the domain $y>0,\ t>0$, satisfying the following boundary conditions:

$$H_\mu(D_x,B_y,D_t)u\big|_{t=0}=g_\mu(x,y) \qquad (\mu=1,2,\dots,m). \tag{2}$$

It is assumed that the coefficients a_k of the operator $\mathcal{L}$ are constant and that the boundary operators H_μ are homogeneous with constant coefficients. In addition, it is assumed that the operator $\mathcal{L}$ is a B-elliptic operator (^{2}) and that the operators $\mathcal{L}$ and H_μ satisfy the Lopatinskii condition (see, for example, (^{1})). We shall seek classical solutions of this problem by means of the Fourier–Bessel transform, i.e. we shall seek solutions in the form

$$u(x,y,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} v(\xi,\sigma,t)e^{ix\xi}j_\nu(\sigma y)\sigma^{2\nu+1}\,d\xi\,d\sigma. \tag{3}$$

Then for v one obtains the boundary-value problem

$$\mathcal{L}(\xi,\sigma,D_t)v = 0, \qquad H_\mu(\xi,\sigma,D_t)v\big|_{t=0} = g_\mu(\xi,\sigma) \tag{4}$$

on the half-line $t > 0$ for an ordinary differential equation with constant coefficients, for fixed ξ,σ. In order to single out the solutions of problem (4) that decrease exponentially as $t \to \infty$, factorization is carried out, as usual. Since the Lopatinskii condition is satisfied, problem (4) is solvable, and its solution is written in the form

$$v(\xi,\sigma,t) = \sum_{k=1}^{m} \tilde{v}_k(\xi,\sigma,t)\,\hat{g}_k(\xi,\sigma), \tag{5}$$

where

$$v_k(\xi,\sigma,t) = \frac{1}{2\pi i}\int_C \frac{N_k(\xi,\sigma,\tau)}{M^+(\xi,\sigma,\tau)}e^{i\tau t}\,d\tau. \tag{6}$$

is a Poisson basis; C is a contour in the upper half-plane of the complex plane, enclosing all zeros of the polynomial M^+ ($\mathcal{L} = a_0M^+M^-$). If in the formula for the solution of our problem we pass from the Fourier-Bessel images to the functions themselves, we obtain

$$u(x,y,t) = \sum_{j=1}^{m}\int_{-\infty}^{\infty}\int_{0}^{\infty} T_{x,y}^{\alpha,\gamma}K_j(x,y,t)\,g_j(\alpha,\gamma)\,\gamma^{2\nu+1}\,d\alpha d\gamma, \tag{7}$$

where the function $K_j(x,y,t)$, called the Poisson kernel, is defined by the relation

$$K_j(x,y,t) = C_\nu\int_{-\infty}^{\infty}\int_{0}^{\infty} \tilde{v}_j(\xi,\sigma,t)e^{ix\xi}j_\nu(\sigma y)\sigma^{2\nu+1}\,d\xi d\sigma. \tag{8}$$

Here and below the constants C_ν have a quite definite value and their particular form is important. In view of their cumbersomeness we do not write them out.

Taking into account the homogeneity of $\tilde{v}_k$ and introducing polar coordinates in the appropriate way, we obtain the following explicit formulas for the Poisson kernels. If $n+1+2\nu+1 > m_j$, where m_j is the order of the operator H_j, then

$$K_j(x,y,t) =$$

$$= C_j \int_0^{\pi} \sin^{2\nu} \varphi \, d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} \frac{\dot{\sigma}^{\,2\nu+1}}{(x\dot{\xi} + y\dot{\sigma}\cos\varphi + t\beta)^{\,n+1+2\nu+1-m_j}} \frac{N_j(\eta,\beta)}{M^+(\eta,\beta)} \, d\omega_\eta,$$

$$\eta = (\dot{\xi}, \dot{\sigma}). \tag{9}$$

If, however, $m_j \geq n + 1 + 2\nu + 1$, then the corresponding integral is divergent. Applying the well-known regularization device (^{1}), in this case we obtain for the Poisson kernels the following two formulas. If $2\nu + 1$ is an integer, then

$$K_j(x, y, t) =$$

$$= C_j \int_0^{\pi} \sin^{2\nu} \varphi \, d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma} + t\beta)^{m_j - n - 1 - 2\nu - 1} \times$$

$$\times \ln \frac{x\dot{\xi} + y\dot{\sigma} + t\beta}{i} \frac{N_j}{M^+} \dot{\sigma}^{\,2\nu+1} \, d\omega_\eta. \tag{10}$$

If, however, $2\nu + 1$ is a fractional number, then the formula for the Poisson kernel has the form

$$K_j(x, y, t) = \tag{11}$$

$$= C_j \int_0^{\pi} \sin^{2\nu} \varphi \, d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma}\cos\varphi + t\beta)^{m_j - n - 1 - 2\nu - 1} \frac{N_j}{M^+} \dot{\sigma}^{\,2\nu+1} \, d\omega_\eta.$$

The Poisson kernels obtained above can be differentiated, for $t > 0$, arbitrarily many times. When boundary operators are applied, singularities may arise. We need to have kernels such that both they themselves and all their derivatives up to a certain order are continuous in the closed half-space. This is achieved, as is known (^{1}), by constructing the so-called adjoint kernels.

Taking into account the formal self-adjointness of the generalized translation operator, with the aid of the adjoint kernels the solution is written in the form

$$u(x, y, t) =$$

$$= C \sum_{j=1}^{m} \int_{-\infty}^{\infty} \int_0^{\infty} \Delta_B^{(\gamma+q)/2} K_{j,q}(s, \tau, t) \, T_{x,y}^{s,\tau} g_j(x, y) \, \tau^{2\nu+1} \, ds \, d\tau. \tag{12}$$

Integrating by parts in formula (12), we obtain, taking into account the commutativity of Δ_B^r and $T_{x,y}^{s,t}$,

$$u(x,y,t) =$$

$$= C\sum_{j=1}^{m}\int_{-\infty}^{\infty}\int_{0}^{\infty} K_{j,q}(s,\tau,t)\, T_{x,y}^{s,t}\Delta_B^{(\gamma+q)/2} g_j(x,y)\,\tau^{2\nu+1}\,ds\,d\tau, \tag{13}$$

where the operator $\Delta_B = \sum \partial^2/\partial x_i^2 + B_y$.

If $2\nu+1$ is an integer (and consequently q is an integer), then the associated kernels have the form

$$K_{j,q}(x,y,t) = \int_0^{\pi} \sin^{2\nu}\varphi\times$$

$$\times\, d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma}\cos\varphi + t\beta)^{m_j+q}\times$$

$$\times\left(\ln\frac{(x\dot{\xi} + y\dot{\sigma}\cos\varphi + t\beta)}{i} + C\right)\frac{N_j}{M^+}\,\dot{\sigma}^{\,2\nu+1}\,d\omega_\eta. \tag{14}$$

If, however, $2\nu+1$ is a fractional number (hence q is fractional), then the associated kernels have the form

$$K_{j,q}(x,y,t) = \tag{15}$$

$$= C\int_0^{\pi}\sin^{2\nu}\varphi\,d\varphi\int_{C_{R,\delta}} d\beta\int_{|\eta|=1}(x\dot{\xi} + y\dot{\sigma}\cos\varphi + t\beta)^{m_j+q}\,\frac{N_j(\eta,\beta)}{M^+(\eta,\beta)}\,\dot{\sigma}^{\,2\nu+1}\,d\omega_\eta.$$

Let g_μ be infinitely differentiable finite functions in the half-space E_{n+1}. Using the results of V. I. Kononenko (3) on the expansion of finite functions into weighted plane waves, it is not difficult to show that formula (13) gives a solution of problem (1)–(2). Each of the kernels $K_{j,q}$ is analytic at all points of the half-space $t \geq 0$, with the exception of the origin. As for the origin, it is a pole of a certain order.

Theorem. The kernels $K_{j,q}(x,y,t)$, for $t \geq 0$ and $(x,y) \neq 0$, are infinitely differentiable functions, and for their derivatives the following estimates hold:

1. If $2\nu+1$ is an integer and $0 \leq s + 2r \leq m_j + q$, then the inequality

$$\left|D_x^s B_y^r K_{j,q}\right| \leq C\left(|x|^2+y^2+t^2\right)^{(m_j+q-s-2r)/2} \times$$

$$\times\left(1+\ln\left(|x|^2+y^2+t^2\right)^{1/2}\right) \tag{16}$$

holds.

If $s+2r \geq m_j+q+1$, then $D_x^s B_y^r K_{j,q}$ is homogeneous of degree $m_j+q-s-2r$, and the logarithmic term in (16) may be omitted.

2. If $2\nu+1$ is a fractional number (in this case q is also fractional), then

$$\left|D_x^s B_y^r K_{j,q}\right| \leq C\left(|x|^2+y^2+t^2\right)^{(m_j+q-s-2r)/2}. \tag{17}$$

Analogous assertions are also valid for the Poisson kernels K_j.

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Note: Figure translations are in progress. See original paper for figures.

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