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# ON POSITIONAL OPERATIVES

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## Abstract

## Full Text

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MATHEMATICS

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# ON POSITIONAL OPERATIVES

(Presented by Academician V. M. Glushkov, 23 II 1968)

In the present note simple positional operatives are studied <sup>(1,2)</sup>. In particular, the theory of completely simple semigroups of Rees–Sushkevich–Clifford <sup>(3,4)</sup> is generalized to this class of algebras.

1. Let  $\mathfrak{P} = \{\rho_k\}_{k=1}^n$  be a family of permutations  $\rho_k$  of the set  $J_{2n-1} = \{1, 2, \dots, 2n-1\}$ , with  $\rho_1 = \rho_n = \varepsilon$  ( $\varepsilon$  is the identity permutation). An operative  $S$  is called a  $\mathfrak{P}$ -operative if, for any  $k \in J_n = \{1, 2, \dots, n\}$ ,  $x_i \in S$ ,

$$x_1 x_2 \cdots x_{k-1} (x_{k x_{k+1}} \cdots x_{k+n-1}) x_{k+n} \cdots x_{2n-1} = x_{\rho_k 1} x_{\rho_k 2} \cdots x_{\rho_k (2n-2)} x_{\rho_k (2n-1)}. \quad (1)$$

$S$  is called a  $\pi$ -operative if, for every  $k \in J_n$ ,  $\rho_k \{k, k+1, \dots, k+n-1\} = \{k, k+1, \dots, k+n-1\}$ . In other words, with a  $\pi$ -operative  $S$  one can associate a family  $\pi = \{\sigma_k, \pi_k\}_{k=1}^n$  of permutations  $\sigma_k, \pi_k$  of the set  $J_n$  such that  $\pi_1 = \pi_n = \sigma_1 = \sigma_n = \varepsilon$ ,  $\sigma_k k = k$ , and for any  $x_i, y_i \in S$

$$x_1 x_2 \cdots x_{k-1} (y_1 y_2 \cdots y_n) x_{k+1} \cdots x_n = x_{\sigma_k 1} x_{\sigma_k 2} \cdots x_{\sigma_k (k-1)} y_{\pi_k 1} \cdots y_{\pi_k n} x_{\sigma_k (k+1)} \cdots x_{\sigma_k n}.$$

A  $\pi$ -operative is called associative if  $\sigma_k = \pi_k = \varepsilon$  for all  $k \in J_n$ ; alternative if  $n$  is odd and all  $\sigma_k = \pi_{2j+1} = \varepsilon$ ,  $\pi_{2j} = \tau$ , where  $\tau i = n - i + 1$ .

A subset  $A$  of an operative  $S$  which is a  $(k)$ -ideal <sup>(1,2)</sup>, for  $k = 1, 2, \dots, n$ , is called an ideal; for  $k = 1$  and  $k = n$ , a two-sided ideal. An operative  $S$  is called  $i$ -simple if it contains no ideals distinct from zero and  $S$  and is not isomorphic to the operative  $V_n = \{a, 0\}$ , where  $0$  is zero and  $a^n = 0$ .

2. **Theorem 1.** *Every  $i$ -simple  $\mathfrak{P}$ -operative  $S$  with zero  $0$  is the union of its two-sided ideals  $A_\gamma$ :  $S = \bigcup_\gamma A_\gamma$ , the number of which  $|\Gamma|$  is a divisor of the number  $n-1$ ;  $A_\gamma \cap A_\delta = \{0\}$  for  $\gamma \neq \delta$ . If  $T$  is an  $i$ -simple  $\mathfrak{P}$ -operative without zero, then  $T$  contains no proper two-sided ideals.*

Let  $a$  be some fixed element of  $S \setminus \{0\}$  and let  $a = a_1 a_2 \cdots a_n$ , where  $a_j \in S$ . Denote by  $A_j$  that one of the two-sided ideals  $A_\gamma$  which contains the element  $a_j$ ; by  $\xi_{ki}$ , the least positive residue of the number  $\rho_k^{-1}(i + k - 1)$  modulo  $n - 1$ .

**Theorem 2.** For any  $k, j \in J_{n-1} = \{1, 2, \dots, n - 1\}$ ,

$$A_k = A_{\xi_{k1}} A_{\xi_{k2}} A_{\xi_{k3}} \cdots A_{\xi_{kn}}, \quad A_{\xi_{k1}} = A_{\xi_{kn}} = A_k,$$

$$A A_{k_2} A_{k_3} \cdots A_{k_n} = \{0\}, \quad \text{if } k_j \neq \xi_{kj}.$$

3. Introduce the following equivalence  $\delta$  on the set  $J_{n-1}$ :  $j\delta k$  if and only if  $A_j = A_k$ .

**Theorem 3.**  $(\xi_j \xi_{ki})\delta(\xi_{\xi_{ki}})$  for any  $i, j, k \in J_n$ ; all equivalence classes of  $\delta$  consist of one and the same number of elements.

Denote by  $l$  the least positive number such that  $A_{l+1} = A_1$  and  $\pi_{l+1} = \varepsilon$ . For associative operatives the equivalence  $\delta$  turns out to be congruence modulo  $l$ . If for an alternative operative  $\delta$  is not congruence

mod  $l$ , then  $A_i$  can be chosen so that  $j\delta k$  if and only if  $j \equiv k \pmod{l}$  or  $j + k \equiv 1 \pmod{l}$ .

4. The theorem below makes it possible to carry over to  $\pi$ -operatives the theory of ideal factors of semigroups <sup>(3,4)</sup>:

**Theorem 4.** Let  $M$  be a minimal ideal\* of a  $\pi$ -operative  $S$ . Then either  $M^n = \{0\}$ , or  $M$  is an  $i$ -simple operative.

5. An  $i$ -simple operative  $S$  is called  $j$ -**simple** if  $S^{k-1} a S^{n-k} \neq \{0\}$  for all  $a \in S \setminus \{0\}$ ,  $k \in J_n$ .

**Theorem 5.** If  $S$  is a  $j$ -simple  $\mathfrak{p}$ -operative, then all  $A_\gamma$  (see item 2) are its minimal two-sided ideals.

The two-sided ideals  $A_\gamma$  of a  $j$ -simple operative  $S$  will be called its components. A  $\pi$ -operative  $S$  is called a  $\pi^*$ -**operative** if  $\pi_k\{1, n\} = \{1, n\}$  for any  $k \in J_n$ . In particular, an associative and an alternative are  $\pi^*$ -operatives.

**Theorem 6.** Every  $i$ -simple  $\pi^*$ -operative is  $j$ -simple.

6. A  $j$ -simple operative  $S$  is called  $c$ -**simple** if each of its components contains a minimal left and a minimal right ideal; a  $c$ -simple  $\mathfrak{p}$ -operative is called  $c_1$ -**simple** ( $c_2$ -**simple**) if it contains exactly one component (respectively, if each of its left or right ideals is a two-sided ideal).

**Theorem 7.** In order that an  $i$ -simple associative be  $c$ -simple, it is necessary and sufficient that it contain a minimal left ideal and a minimal right ideal.

**Theorem 8.** Every  $c$ -simple  $\mathfrak{p}$ -operative without zero is isomorphic to the suboperative of all nonzero elements of some  $c_1$ -simple  $\mathfrak{p}$ -operative with zero.

**Theorem 9.** Every  $c_2$ -simple  $\mathfrak{p}$ -operative is similar <sup>(1,2)</sup> to a nonzero  $c_2$ -simple  $\pi^*$ -operative in which all  $\sigma_k = \varepsilon$ .

7. A subset  $B$  of an operative  $S$  is called its **biideal** if  $BS^{2n-3}B \subseteq B$ .

**Theorem 10.** In order that a  $j$ -simple  $\mathfrak{p}$ -operative be  $c$ -simple, it is necessary and sufficient that each of its components contain at least one minimal biideal.

**Theorem 11.** In order that an  $i$ -simple  $\pi^*$ -operative (in particular, an associative or an alternative) be  $c$ -simple, it is necessary and sufficient that it contain a minimal biideal.

The last criterion is apparently new even for  $n = 2$  (i.e., for semigroups). The theorem below complements one of the results of the articles <sup>(1,2)</sup>.

**Theorem 12.** Every two-sided invertible element of a  $\mathfrak{p}$ -operative is  $(k)$ -invertible for any  $k \in J_n$ . In order that all elements of a  $\mathfrak{p}$ -operative  $S$  be two-sided invertible, it is necessary and sufficient that  $S$  have no biideals distinct from the operative  $S$  itself.

**Theorem 13.** In order that an associative  $S$  be an  $n$ -group, it is necessary and sufficient that it contain no biideals distinct from  $S$ .

For  $n = 2$  the last theorem is given in <sup>(4)</sup>.

8. Let  $\Gamma, I, \Lambda$  be nonempty sets of indices;  $I = \bigcup_{\gamma \in \Gamma} I_\gamma$ ,  $\Lambda = \bigcup_{\gamma \in \Gamma} \Lambda_\gamma$ , where the  $I_\gamma$  (respectively,  $\Lambda_\gamma$ ) are pairwise disjoint subsets of the set  $I$  ( $\Lambda$ ). The set  $B = \bigcup_{\gamma \in \Gamma} I_\gamma \times \Lambda_\gamma$  will be called the **quasidiagonal** of the Cartesian product  $I \times \Lambda$ .

Let, further,  $G_0 = G \cup \{0\}$  be a group with an externally adjoined zero <sup>(3)</sup>. On the Cartesian product  $G_0 \times B$  define the equivalence

$$\sim: (g; i, \lambda) \approx (g'; i', \lambda')$$

if and only if  $(g; i, \lambda) = (g'; i', \lambda')$  or  $g = g' = 0$ . Denote by  $G \circ B$  the quotient set of the Cartesian pro-

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\* The concepts of a minimal ideal, of a chain, etc. are defined as in the theory of semigroups.

of the product  $G_0 \times B$  by the equivalence  $\sim$ . One may assume that  $G \circ B$  consists of all triples  $(g; i, \lambda)$ , where  $g \in G_0$ ,  $(i, \lambda) \in B$ , with all triples  $0 = (0; i, \lambda)$  identified with one another. If  $(i, \lambda)$  is a fixed element of  $B'$  and  $F \subseteq G_0$ , then by  $(F; i, \lambda)$  we denote the set of all elements  $(g; i, \lambda)$ , where  $g \in F$ .

It turns out that for an arbitrary  $c$ -simple  $p$ -operative  $S$  with zero there exist a group  $G$  and a quasidiagonal  $B$  such that  $S = G \circ B$ . In this case  $0$  is the zero of the operative  $S$ , and the set  $(G_0; i, \lambda)$ , for any  $(i, \lambda) \in B$ , is its minimal biideal. If  $S$  is  $c_1$ -simple, then  $B = I \times \Lambda$ .

9. A **matrix** (an  $\Lambda I$ -matrix) over a group with zero  $G_0$  is an arbitrary function  $P$  assigning to each pair of elements  $\lambda \in \Lambda$ ,  $i \in I$  an element  $p_{\lambda i} \in G_0$ .

Let an operative  $S$  be defined on the set  $G \circ B$  (Sec. 8); for any  $k = 1, 2, \dots, n$  denote by  $v_k, v'_k$  a pair of variables, one of which is equal to  $i_k (\in I)$ , the other to  $\lambda_k (\in \Lambda)$ , with  $(i_k, \lambda_k)$  ranging over the whole set  $B$ ,  $v_1 = x_1, v'_n = i_n$ . Suppose that for each  $k \in J_{n-1}$  a regular  $(3, 4)$ -matrix  $p_k(v_k, v'_{k+1})$  over the semigroup  $G_0$  is defined, and for any indices  $\gamma \in \Gamma$ ,  $k \in J_{n-1}$  an automorphism (or inverse automorphism)  $\psi_k^{(\gamma)}$  of the semigroup  $G_0$ . Moreover, if the group  $G$  is not abelian and  $v'_k = i_k, v_k = \lambda_k$  (respectively  $v'_k = \lambda_k, v_k = i_k$ ), then  $\psi_k^{(\gamma)}$ , for any  $\gamma \in \Gamma$ , is an automorphism (respectively an inverse automorphism) of the semigroup  $G_0$ . We shall call  $S$  a **Rees operative** over the group  $G$  if, for any  $(x_j; i_j, \lambda_j) \in S$ ,

$$(x_1; i_1, \lambda_1)(x_2; i_2, \lambda_2) \dots (x_n; i_n, \lambda_n) = (z; i_1, \lambda_n), \quad (2)$$

where, for  $i_1 \in I_\gamma$ ,

$$z = x_1 \cdot p_1(v_1, v'_2) \cdot \psi_2^{(\gamma)} x_2 \cdot p_2(v_2, v'_3) \dots \psi_{n-1}^{(\gamma)} x_{n-1} \cdot p_{n-1}(v_{n-1}, i_n) \cdot x_n.$$

**Theorem 14.** *Every  $c$ -simple  $\pi$ -operative  $S$  is isomorphic to a Rees operative over some group  $G$ . If the group  $G$  is not abelian, then  $S$  is either associative or alternative.*

For  $c$ -simple  $p$ -operatives Theorem 14 is, generally speaking, not true (cf. Sec. 8 and Theorem 9).

10. Let

$$B = \bigcup_{\gamma=1}^t I_\gamma \times \Lambda_\gamma;$$

$G$  be a group;  $P = (p_{\lambda i})$  a regular  $\Lambda I$ -matrix over  $G_0$  for which from  $p_{\lambda i} \neq 0$  and  $\lambda \in \Lambda_\gamma$  it follows that  $i \in I_{\gamma+1}$  (from  $\lambda \in \Lambda_t$  it follows that  $i \in I_1$ );  $a$  an element of  $G$ ;  $\psi$  an automorphism of the semigroup  $G_0$  such that  $\psi^t = \lambda_a^{-1}$ , where  $n-1 = lt$ ,  $\lambda_a x = a^{-1}xa$ . Denote

$$j * \gamma = \left[ \frac{1}{t}(j + \gamma - 2) \right]$$

(here  $[m]$  is the integer part of the number  $m$ ). Next let  $S(G^{(l)}, B, P)$  be the operative defined on the set  $G \circ B$  (Secs. 8, 9) with operation (2), where for  $i_1 \in I_\gamma$

$$z = \psi^{1*\gamma}(x_1 \cdot p_{x_1 i_2}) \cdot \psi^{2*\gamma}(x_2 \cdot p_{x_2 i_3}) \cdot \dots \cdot \psi^{(n-1)*\gamma}(x_{n-1} \cdot p_{x_{n-1} i_n}) \cdot G \cdot X_n.$$

**Theorem 15.**  $S(G^{(l)}, B, P)$  is a  $c$ -simple associative operative with zero. Every  $c$ -simple associative operative with zero is isomorphic to some operative  $S(G^{(l)}, B, P)$ .

From this there easily follows a description of  $c$ -simple associatives without zero (they all turn out to be bundles of mutually isomorphic  $n$ -groups) and, in particular, of  $c$ -simple associatives without proper right ideals <sup>(5)</sup>. The author has found all isomorphisms of  $c$ -simple associatives.

11. We restrict ourselves here to the description only of  $c_1$ -simple alternatives. Let  $n = 2l + 1$ ;  $G$  be a group;  $B = I \times \Lambda$ ;  $a, s, c$  be elements of  $G$ ;  $\psi$  an automorphism;  $\xi$  an inverse automorphism of the semigroup  $G_0$  such that

$$\psi^l = \lambda_a^{-1}, \quad \xi^2 = \lambda_c, \quad \xi\psi = \lambda_{asc}\psi^{l-1}\xi, \quad \psi a = a, \quad \xi a = \left\{ \prod_{j=1}^{l-1} \psi^j(sc) \right\} \cdot asc;$$

$P = (p_{\mu\lambda})$  and  $Q = (q_{ij})$  be, respectively, a regular  $\Lambda\Lambda$ -matrix and an  $II$ -matrix over  $G_0$ , satisfying, for any  $i, j \in I$ ,  $\mu, \lambda \in \Lambda$ , the condi-

to the conditions  $\xi p_{\lambda\mu} = c^{-1}p_{\chi\lambda}$ ,  $\xi q_{ji} = \psi(s \cdot q_{ij})$ . Let, furthermore,  $S(G, B, P, Q, \xi)$  be an operative defined on the set  $G \circ B$  with action (2), where

$$z = \left\{ \prod_{j=1}^l \psi^{j-1}(x_{2j-1} \cdot p_{x_{2j-1}x_{2j}} \cdot \xi x_{2j} \cdot q_{i_{2j}i_{2j+1}}) \right\} \cdot a \cdot x_n.$$

**Theorem 16.**  $S(G, B, P, Q, \xi)$  is a  $c_1$ -simple alternative. Every  $c_1$ -simple alternative is isomorphic to some operative  $S(G, B, P, Q, \xi)$ .

For  $n = 3$  this result was found in papers <sup>(6,7)</sup>.

12. In describing a  $\pi$ -operative of Rees  $S$  over an abelian group  $G$  (see Theorem 14), as in Sec. 11, we shall restrict ourselves only to  $c_1$ -simple operatives (after Secs. 10–11 one may assume that  $S$  is neither associative nor alternative). For such an operative  $S$ , all  $\sigma_k = \varepsilon$ . The action in  $S$  is still determined by formula (2), where

$$\begin{aligned} z &= z' \cdot p_1(x_1, v'_2) \cdot p_2(v_2, v'_3) \cdot \dots \cdot p_{n-1}(v_{n-1}, i_n), \\ z' &= x_1 \cdot \psi_2 x_2 \cdot \psi_3 x_3 \cdot \dots \cdot \psi_{n-1} x_{n-1} \cdot x_n. \end{aligned} \quad (3)$$

Formula (3) defines on the group  $G$  the  $\pi$ -commutator <sup>(1,2)</sup>, and thereby singles out all dependencies among the automorphisms  $\psi_k$ . All matrices  $p_k(v_k, v'_{k+1})$  satisfy one of the following three conditions:

- 1) The matrices  $p_k$  do not depend on the indices  $\lambda \in \Lambda$  (i.e. have the form  $p_k(i_k, i_{k+1})$ , in particular  $p_k(i_k)$ ,  $p_k(i_{k+1})$ , or  $p_k = c$ ).

- 2) The matrices  $p_k$  do not depend on the indices  $i \in I$ .
- 3)  $p_1 = p_1(v'_2)$ ,  $p_{n-1} = p_{n-1}(v_{n-1})$ ; if the matrix  $p_k$  depends on  $i_k$  (on  $\lambda_k$ ), then  $p_{k-1}$  and  $p_{k+1}$  do not depend on  $\lambda_k$  (respectively on  $i_k$ ).

Thus the functions  $p_1, p_2, \dots, p_{n-1}$  depend not on  $2n - 2$  variables  $i_k, \lambda_k$  (see Secs. 9-11), but only on  $n - 1$  variables. One can also reduce the number of the functions  $p_k$  themselves (as in Secs. 10-11); we shall not do this here.

13. Let  $S$  be a  $c_1$ -simple Rees operative (see Secs. 6, 9—in this case  $|\Gamma| = 1$ ); let  $N$  be a normal divisor of the group  $G$  such that  $\psi_{jN} = N$  for every  $j \in J_n$  ( $\psi_j = \psi_j^{(\gamma)}$ ); let  $\sigma_I$  be an equivalence on the set  $I$  such that, for any matrices  $p_k, p_l$  having the form  $p_k = p_k(i, v')$  or  $p_l = p_l(v, i)$ , it follows from  $i\sigma_I i'$  that  $p_k(i, v') \in p_k(i', v')N$  and  $p_l(v, i) \in p_l(v, i')N$ ; analogously an equivalence  $\sigma_\Lambda$  is defined on the set  $\Lambda$ . Let, further,  $\rho = \rho(N, \sigma_I, \sigma_\Lambda)$  be the following equivalence on the operative  $S$ :  $(x; i, \lambda)\rho(y; j, \chi)$  if and only if  $x = y = o$  or  $y \in xN$ ,  $i\sigma_I j$ ,  $\lambda\sigma_\Lambda \chi$ .

It follows from Sec. 9 that  $\rho$  is a congruence of the operative  $S$ .

The converse of this assertion is as follows. If  $\rho$  is a congruence on a  $c_1$ -simple  $\pi$ -operative  $S$ , then an operative of Rees isomorphic to  $S$  can be chosen so that  $\rho$  has the form  $\rho\{N, \sigma_I, \sigma_\Lambda\}$ . We shall not enumerate here the isomorphisms of Rees operatives.

For  $n = 2$  (i.e. for semigroups) this result is contained in paper <sup>(8)</sup> (see also <sup>(4)</sup>). From this there easily follows a description of the homomorphisms of  $\pi$ -operatives all of whose elements are two-sided invertible <sup>(1, 2)</sup>.

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