THE CAUCHY PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

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THE CAUCHY PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

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In the present note a class of pseudodifferential operators is introduced whose symbols, with respect to one of the variables, admit analytic continuation into a complex half-plane. For these operators a calculus is developed, formally similar to the calculus of hypoelliptic pseudodifferential operators ([1]). In contrast to the standard situation, in this calculus pseudodifferential operators are considered not modulo smoothing operators, but modulo operators of small norm. Such a calculus makes it possible, for the pseudodifferential equations considered in the paper, to prove theorems on the existence and uniqueness of the solution of the Cauchy problem. In the case of differential operators these theorems pass into the corresponding theorems ([2]) for strongly correct operators of constant strength. The basic definitions of this note were suggested by the works ([1], [2]). The proofs of the assertions formulated below are readily obtained by modifying the arguments from ([1]-[3]).

1. We shall use two sets of variables $y = (t, x) = (t, x_1, \dots, x_n)$ and $\eta = (\sigma, \xi) = (\sigma, \xi_1, \dots, \xi_n)$, which will be regarded as dual with respect to the bilinear form

$$(y, \eta) = t\sigma + x_1\xi_1 + \dots + x_n\xi_n.$$

If $\alpha = (\alpha_0, \dots, \alpha_n)$ is an integer multi-index, then

$$D_y^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

where

$$D_0 = -i\partial/\partial t, \qquad D_j = -i\partial/\partial x_j, \quad j = 1, \dots, n.$$

Put $\tau = \sigma - i\rho$ and $\eta - i\rho = (\sigma - i\rho, \xi_1, \dots, \xi_n)$.

We shall call **symbols** the functions $a(y; \eta - i\rho)$, defined and continuous for $(y, \eta) \in R^{n+1}$, $\rho \geq \gamma = \gamma(a)$, belonging to C^∞ in y and independent of y for large $|y|$, i.e.

$$a(y; \eta - i\rho) = a(\eta - i\rho), \qquad |y| > Y. \tag{1}$$

Definition 1. Denote by $\mathfrak{R}$ the class of symbols $a(y; \eta - i\rho)$ satisfying three conditions:

1) The function $a(y;\eta - i\rho)$ is analytic in the variable $\tau = \sigma - i\rho$ in the half-plane $\rho \geq \gamma$ and

$$a(y;\eta - i\rho) \neq 0 \quad \text{for } \rho \geq \gamma; \tag{2}$$

2) for any multi-indices α and any $\eta', \eta'' \in R^{n+1}$

$$\left|D_y^\alpha a(y;\eta' - i\rho) - D_y^\alpha a(y;\eta'' - i\rho)\right| < \varepsilon_\alpha(\rho)(1 + |\eta' - \eta''|)^r \left|a(y;\eta'' - i\rho)\right|, \tag{3}$$

where $\varepsilon_\alpha(\rho) \to 0$ as $\rho \to +\infty$;

3) for any $y', y'' \in R^{n+1}$ and any $\alpha \geq 0$

$$\left|D_y^\alpha a(y;\eta - i\rho)\right| \leq c_\alpha \left|a(y'';\eta - i\rho)\right|, \qquad \rho \geq \gamma. \tag{4}$$

Condition 1) is an analogue of I. G. Petrovskii' s correctness condition ([4]), condition 2) generalizes the condition of strong correctness ([2]), and condition 3) is an analogue of the condition of constancy of strength ([2], [5]).

The class of symbols $\mathfrak{R}$ is closed with respect to a number of algebraic operations.

Proposition 1 (cf. ([1]), Lemmas 5.2, 5.3). *The symbols from $\mathfrak{R}$ form a group under multiplication.*

Proposition 2 (cf. ([1]), Lemma 5.4). Let $D \subset C^1$ be the domain of values of the function $a(y;\eta - i\rho)$ for $y \in R^{n+1}$, $\eta \in R^{n+1}$, $\rho \geqslant \gamma$, and suppose that in the domain D one can choose a single-valued branch of the function $\ln z$ ($z \in D$).

Then for any complex λ

$$a(y;\eta - i\rho) \in \mathfrak{R} \Rightarrow a^\lambda(y;\eta - i\rho) = \exp\{\lambda \ln a\} \in \mathfrak{R}.$$

Example. It is not hard to verify that

$$\sigma - i\rho - im(y,\omega)|\xi|^\mu \in \mathfrak{R}, \tag{5}$$

where $\mu \geqslant 1$, $\omega = \xi|\xi|^{-1}$, $\operatorname{Re} m(y,\omega) > m > 0$, and the function $m(y,\omega)$ belongs to C^∞ in y, to C^1 in ω (for $|\omega| = 1$), and does not depend on y for large $|y|$.* By Proposition 1, the product (or quotient) of symbols of type (5) with various μ belongs to $\mathfrak{R}$. By Proposition 2, any (complex) power of the symbol (5) also belongs to $\mathfrak{R}$.

In the case of symbols that depend polynomially on τ, ξ, the condition of belonging to the class $\mathfrak{R}$ is as follows.

Proposition 3. Let $P(y;\tau,\xi)$ be a family of polynomials in the variables (τ,ξ), whose coefficients are C^∞-functions of y, equal to constants for large $|y|$. Suppose that the following conditions are satisfied:

I.

$$|P(y;\eta'-i\rho)-P(y;\eta''-i\rho)| < \varepsilon(\rho)(1+|\eta'-\eta''|)^r|P(y;\eta''-i\rho)|, \quad \varepsilon(\rho)\to 0 \quad \text{as } \rho\to+\infty.$$

II. For all $y', y'' \in R^{n+1}$

$$|P(y';\eta-i\rho)| \leqslant c|P(y'';\eta-i\rho)|, \quad \rho\geqslant\gamma.$$

Then $P(y;\eta-i\rho)\in\mathfrak{R}$.

With the aid of Lemma 3.1.5 from ([5]), it is not hard to show that condition I is equivalent to the conditions of strong correctness from ([2]). Therefore examples of strongly correct polynomials of constant strength from ([2]) are at the same time examples of symbols from $\mathfrak{R}$.

2. Denote by $\tilde{u}(\eta)$ the Fourier transform of the function $u(y)$, and by H^s the space of such distributions u that $(1+|\eta|)^s\tilde{u}(\eta)\in L_2$. By $H_+^{s,\gamma}$ we denote the set of such distributions u that $e^{-\gamma t}u\in H^s$ and

$$\operatorname{supp} u \subset \overline{R_+^{n+1}},$$

where R_+^{n+1} is the half-space $t>0$; $H_+^{\pm\infty,\gamma}$ will denote the intersection (union) of the spaces $H_+^{s,\gamma}$. For functions $u\in H_+^{-\infty,\gamma}$ there exists, for $\rho>\gamma$, the (complex) Fourier transform $\tilde{u}(\sigma-i\rho,\xi)$. On this basis, to each symbol $a(y;\tau,\xi)\in\mathfrak{R}$ we associate the pseudodifferential operator (cf. ([2,3]))

$$^\gamma a(y;D)u(y) = (2\pi)^{-(n+1)/2}\int e^{i(y,\eta-i\rho)}a(y;\eta-i\rho)\tilde{u}(\eta-i\rho)\,d\eta =$$

$$= e^{\rho t}a(y;D_t-i\rho,D_x)e^{-\rho t}u, \quad \rho\geqslant\gamma, \quad u\in H_+^{\infty,\gamma}. \tag{6}$$

Here $a(y;D_t-i\rho,D_x)$ denotes the ordinary pseudodifferential operator in H^s. Shifting the contour of integration in the integral (6), it is not hard to verify that the right-hand side does not depend on ρ when $\rho\geqslant\gamma$.

It can be shown ([3]) that the mapping

$$e^{-\gamma t}\gamma(1+iD_t+|D_x|)^s: \quad H_+^{s,\gamma}\to L_2(R_+^{n+1}) \tag{7}$$

is an isomorphism. If in $H_+^{s,\gamma}$ one defines the norm

$$\|u\|_{s,\gamma}^2 = \int |(1+i(\sigma-i\rho)+|\xi|)^s\tilde{u}(\eta-i\rho)|^2\,d\eta,$$

then this isomorphism becomes isometric.

To each constant symbol $d(\tau,\xi)\in\mathfrak{R}$ we associate the space

$$H_+^{s,\gamma} = \{u\in H_+^{-\infty,\gamma},\ {}^\gamma d(D)u\in H_+^{s,\gamma}\},$$

and define in it the norm

$$\|u\|_{d,s,\gamma} = \|^{\gamma} d(D) u\|_{s,\gamma}.$$

* Symbols of type (5) are called parabolic (see ([6])).

Then the mapping

$$e^{-\gamma t}\gamma(1 + iD_t + |D_x|)^s \, {}^{\gamma}d(D) : \overset{+}{H}{}^{s,\gamma}_{d} \to L_2(R^{n+1}+) \tag{8}$$

will be an isomorphism.

Theorem 1. Let $a(y;\tau,\xi), d(\tau,\xi) \in \mathfrak{R}$. Then the operator

$${}^{\rho}a(y;D) : \overset{+}{H}{}^{s,\rho}_{da(\infty)} \to \overset{+}{H}{}^{s,\rho}_{d}, \qquad a(\infty) = a(\infty;\tau,\xi),\ \rho \geq \gamma,$$

is continuous and its norm does not depend on ρ.

Theorem 2. Let $a(y;\tau,\xi), b(y;\tau,\xi), d(\tau,\xi) \in \mathfrak{R}$ and $c(y;\tau,\xi) = a(y;\tau,\xi)b(y;\tau,\xi)$. Then

$$\|[{}^{\rho}a(y;D){}^{\rho}b(y;D) - {}^{\rho}c(y;D)]u\|_{d,s,\rho} \leq \delta(\rho)\|u\|_{c(\infty)d,s,\rho}; \tag{9}$$

$$\delta(\rho) \to 0 \quad \text{as } \rho \to +\infty.$$

The proof of Theorems 1 and 2, with the help of the isomorphisms (7), (8) and the Fourier transform, reduces to estimating certain integral operators in L_2 (see ([1–3])).

A consequence of Theorems 1 and 2 is

Theorem 3. If $a(y;\tau,\xi) \in \mathfrak{R}$ and $a(\infty) = a(\infty;\tau,\xi)$, then for large ρ the equation

$${}^{\rho}a(y;D)u = f(y) \in \overset{+}{H}{}^{s,\rho} \tag{10}$$

has a unique solution $u(y) \in \overset{+}{H}{}^{s,\rho}_{a(\infty)}$.*

Proof. We shall seek the solution of equation (10) in the form

$$u = {}^{\rho}a^{-1}(y;D)g, \quad \text{where } g \in \overset{+}{H}{}^{s,\rho}.$$

Substituting in (10), we obtain for g the equation

$$g + Tg = g + [{}^{\rho}a(y;D)^{\rho}a^{-1}(y;D) - 1]g = f. \qquad (11)$$

By Theorems 1 and 2, the operator T maps $\overset{+}{H}{}^{s,\rho}$ into itself and, for large ρ, its norm is less than 1. Hence it follows that there exists $g \in \overset{+}{H}{}^{s,\rho}$, and consequently also a solution $u \in \overset{+}{H}{}^{s,\rho}_{a(\infty)}$ of equation (10).

To prove uniqueness, one must apply the operator ${}^{\rho}a^{-1}(y;D)$ to both sides of (10). If ρ is so large that the norm (in $\overset{+}{H}{}^{s,\rho}_{a(\infty)}$) of the operator ${}^{\rho}a^{-1}(y;D)^{\rho}a(y;D) - 1$ is less than 1/2, then the estimate

$$\|u\|_{a(\infty),s,\rho} \leq 2\|{}^{\rho}a^{-1}(y;D)f\|_{a(\infty),s,\rho} \leq C\|f\|_{s;\rho}$$

holds, from which uniqueness for equation (10) follows.

3. Hörmander ([7]) introduced a class of (differential) hypoelliptic operators for which conditions of type (4) are not satisfied. In ([8,9]) the results of ([7]) were extended to the case of pseudodifferential operators, and in ([8]) an interesting technique was developed using a partition of unity in Fourier images. The methods of ([8]) make it possible to study the Cauchy problem for certain pseudodifferential operators of "variable order."

Definition 2. Denote by $\mathfrak{R}_{\varepsilon,\delta}$ the class of symbols $a(y;\tau,\xi)$ belonging to C^{∞} in all variables, analytic in τ in the half-plane $\operatorname{Im}\tau \leq -\gamma$, independent of y for large $|y|$, and satisfying two conditions:

1) there exist numbers m_1, m_2, c_1, c_2 such that

$$c_1(1 + |\tau| + |\xi|)^{m_1} < |a(y;\tau,\xi)| < c_2(1 + |\tau| + |\xi|)^{m_2}, \qquad \operatorname{Im}\tau \leq -\gamma;$$

2) there exist ε, δ, $0 \leq \delta < \varepsilon \leq 1$, such that for all multi-indices α, β

$$|D_{\eta}^{\alpha} D_{y}^{\beta} a(y;\tau,\xi)| < c_{\alpha\beta}(1 + |\tau| + |\xi|)^{-\varepsilon|\alpha| + \delta|\beta|}|a(y;\tau,\xi)|.$$

* The function $u(y)$ is naturally called the solution of the homogeneous Cauchy problem in the half-space $t \geq 0$ for the operator ${}^{\gamma}a(y;D)$.

As an example of a symbol from $\mathfrak{L}_{\varepsilon,\delta}$, we indicate $(\tau - im(x,\omega)|\xi|^{2\mu})^{\lambda(y)}$, where $\omega = \xi|\xi|^{-1}$, $\operatorname{Re} m(x,\omega) > m > 0$, $\mu > 0$ is an integer, and $\lambda(y)$ is a complex C^{∞} function, $\lambda(y) = \lambda$ for $|y| > Y$. Let us also note that $\mathfrak{L}_{\varepsilon,0} \subset \mathfrak{R}$.

Theorem 4 (cf. (8), Theorem 5.4). If $a(y;\tau,\xi) \in \mathfrak{L}_{\varepsilon,\delta}$, then for large ρ equation (10) has a unique solution $u \in \overset{+}{H}{}^{-\infty,\rho}$, and moreover

$$(D_\eta^\alpha D_y^\beta a)(y;D)u \in \overset{+}{H}{}^{s+\varepsilon|\alpha|-\delta|\beta|,\gamma}.$$

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Note: Figure translations are in progress. See original paper for figures.