INTERPOLATION OF FUNCTIONS WITH A FINITE NUMBER OF SINGULARITIES BY MEANS OF RATIONAL FUNCTIONS

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Abstract

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INTERPOLATION OF FUNCTIONS WITH A FINITE NUMBER OF SINGULARITIES BY MEANS OF RATIONAL FUNCTIONS

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Let an analytic function $f(z)$ in the entire complex plane have only a few (a finite number of) singularities $\alpha_1,\ \alpha_2, \dots, \alpha_p$ $(p > 1)$; we shall interpolate it by means of rational functions, and it is necessary to assume that: 1) each singular point of the function $f(z)$ being interpolated must at the same time be a singular point of the interpolating rational function; 2) each singular point $\alpha_1, \alpha_2, \dots, \alpha_p$ $(p > 1)$ of the function $f(z)$ being interpolated must be a limit point of the sequence $\{x_n\}$ of interpolation nodes; 3) the points x_m $(m = 0, 1, 2, \dots)$ are distinct and different from the singularities of the function $f(z)$, i.e., $x_m \neq a_k$ $(k = 1, 2, \dots, p;\ m = 0, 1, 2, \dots, n, \dots)$.

It is known that there exists one and only one rational function of degree n (see (6), p. 226) of the form

$$r_n(z) = P_n(z)/(z-\alpha_1)^{\mu_1}(z-\alpha_2)^{\mu_2}\cdots(z-\alpha_p)^{\mu_p}, \tag{1}$$

($P_n(z)$ is a polynomial of degree n, $\mu_\nu,\ \nu = 1, 2, \dots, p$, are nonnegative integers, $\sum_{\nu=1}^{p} \mu_\nu = n$), satisfying the conditions

$$r(x_m) = f(x_m) \quad (m = 0, 1, 2, \dots). \tag{2}$$

We shall call this interpolation process uniformly convergent in a certain domain D if the equality $\lim_{n\to\infty} r_n(z) = f(z)$ $(z \in D)$ holds.

In the present paper we find sufficient conditions for convergence of the indicated interpolation process, when the degree of the rational function by means of which the interpolation is performed increases without bound, in terms of the maximum of the modulus of the function being interpolated and the density function of the sequence of interpolation nodes*.

First of all, note that the remainder term of the interpolation, i.e., the difference $R_n(z) = f(z) - r_n(z)$, can be represented in the form

$$R_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \frac{(\zeta - \alpha_1)^{\mu_1} \cdots (\zeta - \alpha_p)^{\mu_p}}{(z - \alpha_1)^{\mu_1} \cdots (z - \alpha_p)^{\mu_p}} \frac{\Pi_n(z)}{\Pi_n(\zeta)} \, d\zeta, \tag{3}$$

where we put $\Pi_n(z) = \prod_{m=0}^{n}(z - x_m)$, and the closed contour (C) encloses the point z and all the points x_m $(m = 0, 1, \dots, n)$, and does not enclose any of the points α_ν $(\nu = 1, 2, \dots, p)$.

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* A similar problem, in particular in the class of all analytic functions having only one singular point at infinity, i.e., in the class of all entire functions, was considered for the interpolation process of Newton ([1]), Abel-Goncharov ([2]), and trigonometric interpolation ([3]).

Further, $R_n(z)$ is represented in the form

$$R_n(z) = \sum_{\nu=1}^{p} R_{n,\nu}(z),$$

where $R_{n,\nu}(z)$ are integrals with the same integrand as integral (3), but taken in the opposite direction along closed contours (γ_ν), enclosing the point a_ν and not enclosing any of the points

$$a_1, \dots, a_{\nu-1}, a_{\nu+1}, \dots, a_p, z, x_0, x_1, \dots, x_n.$$

Bearing in mind that the sequence of interpolation nodes $\{x_n\}$ has no limit points other than a_ν $(\nu = 1, 2, \dots, p)$, the set of numbers x_n can be divided into p sequences (S_ν) $(\nu = 1, 2, \dots, p)$, each of which has as its limit one of the numbers a_ν:

$$(S_\nu) : x_1^{(\nu)}, x_2^{(\nu)}, \dots, x_n^{(\nu)}, \dots; \qquad \lim_{n \to 0} x_n^{(\nu)} = a_\nu \quad (\nu = 1, 2, \dots, p).$$

Thus, by directly estimating all the integrals $R_{n,\nu}(z)$ $(\nu = 1, 2, \dots, p)$, we arrive at the following conclusion.

Theorem 1. Let $f(z)$ be an analytic function having only a finite number of singular points $a_1, a_2, \dots, a_p$ in the whole z-plane, with maximum modulus in a neighborhood of each singular point

$$M_\nu(f; r^{(\nu)}) = \max_{|z-a_\nu| \le r^{(\nu)}} |f(z)| \quad (\nu = 1, 2, \dots, p).$$

Further, let the numbers $a_1, a_2, \dots, a_p$, and only these numbers, be limit points of the set of numbers x_n $(n = 0, 1, 2, \dots)$, which is divided into p sequences $\{x_m^{(k)}\}$ $(k = 1, 2, \dots, p)$, converging respectively to the numbers a_k $(k = 1, 2, \dots, p)$.

Finally, let $n_\nu(r^{(\nu)})$ denote the number of terms of the subsequence $\{x_m^{(\nu)}\}$ lying outside the circle $|z-a_\nu| \le r^{(\nu)}$ $(\nu = 1, 2, \dots, p)$.

Then the Newton-type interpolation process by rational functions $r_n(z)$ converges uniformly to the function $f(z)$ in every finite domain

$$G(|z-a_\nu| \ge \eta > 0,\ |z| \le \rho),$$

if the inequalities*

$$\log M_\nu(f; \theta r^{(\nu)}) < c(\theta) n_\nu(r^{(\nu)}) \quad (\nu = 1, 2, \dots, p),$$

are satisfied, where

$$c(\theta) < \log(1-\theta)/\theta, \qquad 0 < \theta < \tfrac{1}{2}.$$

We shall call the order and type of the analytic function $f(z)$ near the singular point $z = a_\nu$ $(\nu = 1, 2, \dots, p)$, respectively, the numbers ρ_ν and A_ν, defined by the relations

$$\overline{\lim_{z\to a_\nu}} \frac{\log\log|f(z)|}{\log|1/(z-a_\nu)|} = \rho_\nu \quad (\rho_\nu \ge 0),$$

$$\overline{\lim_{z\to a_\nu}} |z-a_\nu|^{\rho_\nu} \log|f(z)| = A_\nu \quad (\rho_\nu > 0,\ A_\nu > 0),$$

where the upper limits are taken with respect to possible paths leading to the point a_ν.

Further, we shall call the order and type of convergence of the set of interpolation points $\{x_n\}$ near its limit point a_ν the numbers ρ'_ν and ω_ν, defined by the relations

* In the case when the function $f(z)$ has only one singular point $(p = 1)$, it may be placed at infinity, i.e., the function may be regarded as entire, and the interpolation may be carried out by means of polynomials, i.e., by rational functions with poles at infinity. In this case Theorem 1 coincides with the main theorem of the work ([1]), which, as was shown, cannot be improved, i.e., θ cannot be taken from the interval $\theta > \frac{1}{2}$.

$$\frac{1}{\rho_\nu} = \overline{\lim_{n\to\infty}} \frac{\log|1/(x_n^{(\nu)} - \alpha_\nu)|}{\log n} \qquad (\nu = 1, 2, \dots, p),$$

$$\omega_\nu = \overline{\lim_{n\to\infty}} \frac{1}{n^{1/\rho_\nu}|x_n^{(\nu)} - \alpha_\nu|} \qquad (\nu = 1, 2, \dots, p).$$

From Theorem 1 one can obtain a number of concrete corollaries.

Corollary 1. *The interpolation process by rational functions converges uniformly in every closed domain containing no singular points of the interpolated function, provided that near each point the order of growth of the function is smaller than the order of convergence of the interpolation points*, i.e. $\rho'_\nu > \rho_\nu$ $(\nu = 1, 2, \dots, p)$.

Finally, consider the case where $\rho'_\nu = \rho_\nu$ $(\nu = 1, 2, \dots, p)$.

Corollary 2. *If the order of growth of the analytic function $f(z)$ near each of its singular points coincides respectively with the order of convergence of the interpolation points*, i.e. $\rho'_\nu = \rho_\nu$ $(\nu = 1, 2, \dots, p)$, *then the interpolation process by rational functions converges uniformly in every closed domain containing no singular points of the interpolated function, provided that the types of growth of the function A_ν and the types of convergence of the interpolation points ω_ν satisfy the inequalities*

$$A_\nu < \omega_\nu^{-\rho_\nu} \frac{1}{\rho_\nu} \frac{\lambda^{\rho_\nu}}{(1+\lambda)^{\rho_\nu - 1}} \qquad (\nu = 1, 2, \dots, p), \tag{5}$$

where λ is the positive root of the equation $\lambda e^{1+\lambda} = 1$.

This assertion differs from the corresponding theorem of V. L. Goncharov (5) in that, instead of the last inequality, Goncharov obtained the inequality

$$A_\nu < B_\nu = \begin{cases} \omega_\nu^{-\rho_\nu} h_{\rho_\nu}(2), & \text{if } \varkappa_\nu \geqslant 2^{-\rho_\nu}, \\ \omega_\nu^{-\rho_\nu} h_{\rho_\nu}(\varkappa_\nu^{-1/\rho_\nu}), & \text{if } \varkappa_\nu \leqslant 2^{-\rho_\nu}, \end{cases} \tag{6}$$

where

$$h_q(u) = q \int_u^\infty \log(t-1)\, \frac{dt}{t^{q+1}} \qquad (q > 0,\ u > 1).$$

It can be shown that the right-hand side of inequality (5) does not exceed the right-hand side of inequality (6).

We note that in the case where $f(z)$ has only one singular point $z = \alpha$, inequality (6) takes the form

$$A < B = \omega^{-\rho} h_\rho(2). \tag{7}$$

Under the assumption that the point α is the point at infinity, inequality (7) was established by A. O. Gelfond (4), and it was proved that the constant appearing on the right-hand side of (7) cannot be increased.

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CITED LITERATURE

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4. A. O. Gelfond, *Atti Acad. Lincei*, **11**, 377 (1930).

5. V. L. Goncharov, *Izv. AN SSSR, ser. matem.*, **1**, 171 (1937).

6. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, IL, 1961.

* This assertion was first proved directly by V. L. Goncharov (5).

Note: Figure translations are in progress. See original paper for figures.