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ASYMPTOTICS OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR THE BIHARMONIC EQUATION

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The purpose of this note is to justify an asymptotic method for the problem of the pure cylindrical bending of a thin simply supported flat plate.

After simple transformations, the construction of the asymptotics of the solution of this problem reduces to the following problem:

Let $\Omega = \{(x, y) : \ 0 \leq x \leq \pi, \ -h \leq y \leq +h\}$ be a rectangle of height $2h$, sufficiently small in comparison with the base. In Ω the following problem is considered:

$$u_{yyyy} + 2u_{yyxx} + u_{xxxx} = 0; \tag{1}$$

$$\left[u_{yy} + 2u_{xx}\right]_{t=\pm h} = \pm p/2, \qquad \left[u_{yyy} + 2u_{yxx}\right]_{t=\pm h} = 0; \tag{2}$$

$$u_x\big|_{x=0,\ x=\pi} = 0, \qquad u_{xxx}\big|_{x=0,\ x=\pi} = 0, \tag{3}$$

where $p(x)$ is a given smooth function.

To construct the asymptotics of the solution of problem (1)–(3) with respect to the small parameter h, make the change of variables $y = th$. In the new coordinates (1) and (2) are written as

$$\Delta^2 u \equiv h^{-4} u_{tttt} + 2h^{-2} u_{ttxx} + u_{xxxx} = 0; \tag{4}$$

$$\left[h^{-2} u_{tt} + 2u_{xx}\right]_{t=\pm 1} = \pm p/2, \qquad \left[h^{-2} u_{ttt} + 2u_{txx}\right]_{t=\pm 1} = 0. \tag{5}$$

To the splitting of the operator $\Delta^2 u$ in (4) there corresponds an iterative process, if the approximate solution of (4) is sought in the form

$$u = h^{-2}u^{-2} + h^{-1}u^{-1} + u^0 + hu^1 + \cdots. \tag{6}$$

Substituting the expression for the function u from (6) into (4), (5) and comparing the coefficients of equal powers of h, we obtain

$$u^{-2}_{ttt} = 0, \qquad u^{-2}_{tt}\big|_{t=\pm 1} = 0, \qquad u^{-2}_{ttt}\big|_{t=\pm 1} = 0.$$

Consequently, $u^{-2}(x,t) = u^{-2,0}(x) + tu^{-2,1}(x)$, where $u^{-2,0}(x)$, $u^{-2,1}(x)$ and all subsequent $u^{i,k}(x)$ $\{i = -1, 0, 1, \dots,\ k = 0, 1\}$ are arbitrary functions to be determined. Similarly

$$u^{-1}(x,t) = u^{-1,0}(x) + tu^{-1,1}(x).$$

Further,

$$u^0_{ttt} = 0, \qquad u^0_{tt}\big|_{t=\pm 1} = -2u^{-2,0}_{xx}, \mp 2u^{-2,1}_{xx}, \qquad u^0_{ttt}\big|_{t=\pm 1} = -2u^{-2,1}_{xx}.$$

Hence

$$u^0(x,t) = u^{0,0}(x) + tu^{0,1}(x) - t^2 u^{-2,0}_{xx} - \frac{1}{3}t^3 u^{-2,1}_{xx}.$$

In exactly the same way,

$$u^1(x,t) = u^{1,0}(x) + tu^{1,1}(x) - t^2 u^{-1,0}_{xx} - \frac{1}{3}t^3 u^{-1,1}_{xx}.$$

Comparing the coefficients at h^{-2}, we obtain

$$u^2_{tttt} = 3u^{-2,0}_{xxxx} + 3tu^{-2,1}_{xxxx}; \tag{7}$$

$$u^2_{tt}\big|_{t=+1} = p/2 - 2u^{0,0}_{xx} - 2u^{0,1}_{xx} + 2u^{-2,0}_{xxxx} + \frac{2}{3}u^{-2,1}_{xxxx}; \tag{8}$$

$$u^2_{tt}\big|_{t=-1} = -p/2 - 2u^{0,0}_{xx} + 2u^{0,1}_{xx} + 2u^{-2,0}_{xxxx} - \frac{2}{3}u^{-2,1}_{xxxx}; \tag{9}$$

$$u^2_{ttt}\big|_{t=+1} = -2u^{0,1}_{xx} + 4u^{-2,0}_{xxxx} + 2u^{-2,1}_{xxxx}; \tag{10}$$

$$u^2_{ttt}\big|_{t=-1} = -2u^{0,1}_{xx} - 4u^{-2,0}_{xxxx} + 2u^{-2,1}_{xxxx}. \tag{11}$$

Obviously, the adjoint homogeneous problem corresponding to problem (7)–(11) has a nonzero solution of the form $w = c_1 + tc_2$. Therefore, in order that problem (7)–(11) have a solution, it is necessary and sufficient that the condition

$$\int_{-1}^{+1} u^2_{tttt} w\, dt = \left[u^2_{ttt} w - u^2_{tt} w_t\right]_{t=-1}^{t=+1}. \tag{12}$$

be satisfied.

Condition (12) is called the solvability condition for problem (7)–(11). It is obtained from Green' s formula, if the boundary conditions for w are taken into account.

Substituting the expressions u^2_{tttt}, u^2_{tt}, w, and w_t from (7)–(11) into (12), after simplification we obtain:

$$u^{-2,0}_{xxxx} = a^{-2,0} p(x); \qquad u^{-2,1}_{xxxx} = a^{-2,1} p(x),$$

where $a^{-2,0}$ and $a^{-2,1}$ are known constants.

Continuing the process, we find

$$u^k_{tttt} = -2u^{k-2}_{ttxx} - u^{k-4}_{xxxx}, \qquad k = 3, 4, 5, \dots; \tag{13}$$

$$u^k_{tt}\big|_{t=\pm 1} = -\,2u^{k-2}_{xx}\big|_{t=\pm 1}, \qquad u^k_{ttt}\big|_{t=\pm 1} = -\,2u^{k-2}_{txx}\big|_{t=\pm 1}. \tag{14}$$

Substituting the expressions found for the functions $u^{k-2}(x,t)$ and $u^{k-4}(x,t)$ into problem (13), (14), and writing the solvability condition for the resulting problem, we find that the unknown functions $u^{k,0}(x)$ and $u^{k,1}(x)$ ($k = -1, 0, 1, \dots$) satisfy an ordinary differential equation of fourth order. We shall determine these functions from the following problems:

$$u^{k,0}_{xxxx} = f^{k,0}(x), \quad u^{k,0}(0) = \alpha^k_1, \quad u^{k,0}(\pi) = \alpha^k_2, \quad u^{k,0}_x(0) = \alpha^k_3, \quad u^{k,0}_x(\pi) = \alpha^k_4; \tag{15}$$

$$u^{k,1}_{xxxx} = f^{k,1}(x), \quad u^{k,1}(0) = \beta^k_1, \quad u^{k,1}(\pi) = \beta^k_2, \quad u^{k,1}_x(0) = \beta^k_3, \quad u^{k,1}_x(\pi) = \beta^k_4;$$

$$k = -2, -1, 0, 1, \dots, \tag{16}$$

where $f^{k,i}(x)$, α_j^k, and β_j^k are known. Thus we determine all the functions entering into expansion (6).

Obviously, the functions $u^k(x,t)$, generally speaking, do not satisfy the boundary conditions (3). Therefore, to the functions $u^k(x,t)$ we add functions $v^{0,k}(\tau,t)$, $v^{\pi,k}(\eta,t)$ of boundary-layer type so that the resulting sum $u^k+v^{0,k}+v^{\pi,k}$ satisfies all the boundary conditions. The functions $v^{0,k}$ and $v^{\pi,k}$ are determined by a second iteration process.

In order to describe the second iteration process near $x=0$, in a small neighborhood of this boundary we introduce the local coordinates $x=\tau h$, $y=th$. Writing (1), (2), and the boundary conditions at $x=0$ in the new coordinates, we seek an approximate solution of the obtained equation $\Delta^2 v=0$ in the form

$$v^0(\tau,t)=h^{-2}v^{0,-2}+h^{-1}v^{0,-1}+v^{0,0}+hv^{0,1}+... \qquad (17)$$

so that the sum $u+v^0$ satisfies (2) and the boundary conditions at $\tau=0$ $(x=0)$. Substituting the functions v^0 from (17) into the equation $\Delta^2v^0=0$ and the expression for the sum $u+v^0$ into the boundary conditions, and comparing coefficients of like powers of h, to determine the functions $v^{0,k}$ we obtain the problems

$$v^{0,k}_{tttt}+2v^{0,k}_{tt\tau\tau}+v^{0,k}_{\tau\tau\tau\tau}=0; \qquad (18)$$

$$[v^0_{tt}+2v^0_{\tau\tau}]_{t=\pm1}=0, \qquad [v^0_{ttt}+2v^0_{t\tau\tau}]_{t=\pm1}=0; \qquad (19)$$

$$v^{0,k}_{\tau}\big|_{\tau=0}=\varphi^k(t), \qquad v^{0,k}_{\tau\tau\tau}\big|_{\tau=0}=\psi^k(t) \quad (\varphi^k=-u^k_x(0,t), \quad \psi^k=-u^k_{xxx}(0,t)); \qquad (20)$$

$$k=-2,-1,0,1,...$$

We shall show that, under certain conditions on the boundary functions, problem (18)–(20) has a solution that decreases exponentially at infinity with respect to τ.

Denote by $G^0(\tau,t;\xi,\eta)$ the Green' s function of problem (18)–(20) (here and below the index k is omitted). Then, evidently,

$$G^0(\tau,t;\xi,\eta)=G(\tau,t;\xi,\eta)+G(-\tau,t;\xi,\eta),$$

where $G(\tau,t;\xi,\eta)$ is determined from the problem

$$\Delta^2G=\delta(\tau-\xi,t-\eta); \qquad (21)$$

$$[G_{tt} + 2G_{\tau\tau}]_{t=\pm 1} = 0, \qquad [G_{ttt} + 2G_{t\tau\tau}]_{t=\pm 1} = 0 \tag{22}$$

in the class of functions decreasing with respect to τ, as $\tau \to \pm\infty$.

We construct $G(\tau, t; \xi, \eta)$. Let

$$K^0(\lambda, t; \xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda\tau} G(\tau, t; \xi, \eta)\, d\tau. \tag{23}$$

Applying the transform (23) to problem (21) and (22), we obtain

$$K_{tttt} - 2\lambda^2 K_{tt} + \lambda^4 K = \delta(t - \eta); \tag{24}$$

$$[K_{tt} - 2\lambda^2 K]_{t=\pm 1} = 0, \qquad [K_{ttt} - 2\lambda^2 K_t]_{t=\pm 1} = 0, \tag{25}$$

where $K^0(\lambda, t; \xi, \eta) = e^{-i\lambda\xi} K(\lambda, t, \eta)$. It is known that $K(\lambda, t, \eta)$ is a meromorphic function.

By direct computation one can verify that the poles of $K(\lambda, t, \eta)$ coincide with the roots of the equation

$$\Delta(\lambda) = e^{4\lambda} + e^{-4\lambda} - 16\lambda^2 - 2 = 0.$$

It is not difficult to see that, outside four arbitrarily narrow sectors

$$(2k+1)\pi/4 - \varepsilon \le \arg\lambda \le (2k+1)\pi/4 + \varepsilon,$$

there is only a finite number of zeros of $\Delta(\lambda)$. Obviously, the equation $\Delta(\lambda) = 0$ has no roots on the real axis, except $\lambda = 0$, and $\lambda = 0$ is a fourth-order root. Then there exists a strip

$$\Omega' = \{\, |\operatorname{Im}\lambda| \le \sigma,\ \sigma > 0 \,\},$$

where the function $K(\lambda, t, \eta)$, apart from zero, has no poles. It is not difficult to show that

$$\begin{aligned} K &= \frac{1}{2}\{\lambda^{-4}(1 - 3t\eta) + \lambda^{-2}(1/3 + 3t\eta/4 + t^2 - t^2\eta)\} + K^2(\lambda, t, \eta) \\ &= K^1(\lambda, t, \eta) + K^2(\lambda, t, \eta), \end{aligned}$$

where $K^2(\lambda, t, \eta)$ is an analytic function in the strip Ω'.

Taking into account the expression for $K(\lambda, t, \eta)$ and formula (23), we find

$$G^0(\tau, t; \xi, \eta) = 2\int_{-\infty}^{+\infty} \{K^1(\lambda, t, \eta) + K^2(\lambda, t, \eta)\} e^{i\lambda\xi} \cos\lambda\tau\, d\lambda.$$

It is easy to show that the function $K^2(\lambda, t, \eta)$ in the domain of analyticity has the estimate

$$|K^2(\lambda, t, \eta)| \le \mathrm{const}/|\lambda|^3. \tag{26}$$

Using (26), one can show that the function

$$\int_{-\infty}^{+\infty} K^2(\lambda, t, \eta) e^{i\lambda\xi} \cos \lambda\tau \, d\lambda$$

decreases exponentially with respect to τ.

Theorem 1. In order that the functions $v^{0,k}(\tau, t)$ be exponentially decreasing in τ, it is necessary and sufficient that

$$\int_{-}^{+} t^i \varphi^k(t) \, dt = 0, \qquad \int_{-1}^{+1} t^i \psi^k(t) \, dt = 0, \qquad i = 0, 1; \;\; k = -2, -1, 0, 1, \ldots \tag{27}$$

Obviously, conditions (27) can always be ensured by specifying the boundary conditions in the problems (15) and (16) in the corresponding way. We note that conditions (27) are nothing other than the Saint-Venant principle for the problem under consideration.

In an analogous manner one constructs a function of boundary-layer type $v^{\pi}(\eta, t)$ near the boundary $x = \pi$. We multiply $v^{0,k}(\tau, t)$ and $v^{\pi,k}(\eta, t)$ by smoothing functions, and denote the functions thus obtained again by $v^{0,k}(\tau, t)$ and $v^{\pi,k}(\eta, t)$, respectively. Thus, we have found that the asymptotic representation of the solution of problem (1)–(3) has the form:

$$u = \sum_{i=-2}^{N+4} h^i u^i + \sum_{j=-2}^{N} h^j (v^{0,j} + v^{\pi,j}) + h^{N+1} z. \tag{28}$$

Let us estimate z, which is evidently the solution of the following problem:

$$h^{-4} z_{tttt} + 2h^{-2} z_{ttxx} + z_{xxxx} = F; \tag{29}$$

$$\left[h^{-2} z_{tt} + 2z_{xx}\right]_{t=\pm 1} = 0, \qquad \left[h^{-2} z_{ttt} + 2z_{txx}\right]_{t=\pm 1} = 0; \tag{30}$$

$$z_x\big|_{x=0,\ x=\pi} = 0, \qquad z_{xxx}\big|_{x=0,\ x=\pi} = 0, \tag{31}$$

where F is a known function.

Theorem 2. For the solution of problem (29)–(31) the estimate

$$\|z\|_1 \leq \alpha \|F\|_2$$

holds, where α is a constant independent of h; $\| \|_1$ is the norm in the space C; $\| \|_2$ is the norm in the space L_1.

From what has been set forth it follows

Theorem 3. If the function $p(x)$ has derivatives up to order $(N+2)$ inclusive, and the derivatives of orders N and $(N+1)$ vanish at $x=0$ and $x=\pi$, then for the solution of problem (1)–(3) there exists the asymptotic expansion (28), where the functions u^i are determined by the first iterative process; $v^{0,i}$, $v^{\pi,i}$ are functions of boundary-layer type and are determined by the second iterative process; and $h^{N+1}z$ is the remainder term, which tends to zero as $h \to 0$ like h^{N+1} in the metric of the space C.

Remark. From what has been set forth it follows, in particular, that the set of eigenfunctions of the spectral problem

$$y^{IV} + 2\lambda^2 y'' + \lambda^4 y = 0, \qquad \left[y'' + 2\lambda^2 y\right]_{x=0,\ x=1} = 0,$$

$$\left[y''' + 2\lambda^2 y'\right]_{x=0,\ x=1} = 0,$$

corresponding to eigenvalues with $\operatorname{Re}\lambda < 0$, forms a basis in the space $C(0,1)$.

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Note: Figure translations are in progress. See original paper for figures.