ON INSTABILITY UNDER CONSTANTLY ACTING PERTURBATIONS

1968

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

M. M. KHAPAEV

ON INSTABILITY UNDER CONSTANTLY ACTING PERTURBATIONS

(Presented by Academician A. N. Tikhonov on 10 III 1967)

The subject of investigation in the present work is systems of ordinary differential equations containing constantly acting perturbations μR_i, where μ is a small parameter,

$$dx_i/dt = f_i(t, x_1, \dots, x_n) + \mu R_i(t, x_1, \dots, x_n), \qquad 1 \le i \le n. \tag{1}$$

If the small parameter $\mu = 0$, we obtain the system

$$dx_i/dt = f_i(t, x_1, \dots, x_n), \qquad 1 \le i \le n. \tag{2}$$

We shall assume that the functions f_i in the region

$$|x_i| \le h, \qquad t \ge 0 \tag{3}$$

are continuous and with respect to the variables x_i satisfy a Lipschitz condition with constant N; moreover, $f_i(t, 0, \dots, 0) = 0$.

In the article conditions are established for instability of the equilibrium point, determined by the properties of the perturbations μR_i. In contrast to the known theorems of Lyapunov' s second method and Chetaev' s theorems on instability ([1],[2]), here it is required that the derivative of the corresponding function v, computed by virtue of the equations of system (2), be nonnegative

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i \ge 0. \tag{4}$$

It is also assumed that there exists the mean

$$\varphi_0(t_0, x_{10}, \dots, x_{n0}) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i(t, \bar{x}_1, \dots, \bar{x}_n)\, dt, \tag{5}$$

where the integral is computed along the integral curve $\bar{x}_i(t)$ of system (2) with initial conditions $\bar{x}_i(t_0) = x_{i0}$, and the passage to the limit is performed uniformly with respect to t_0 and $x_{10}, \dots, x_{n0}$.

We shall assume that the following smoothness conditions are fulfilled:

$$\text{the derivatives } \partial v/\partial x_i \text{ and the perturbations } R_i \tag{6}$$

are continuous in x_i for $|x_i| \leq h$ uniformly with respect to t, although these conditions, as was indicated in paper (³), are easy to weaken.

Theorem 1. *Suppose that there exists $v(t, x_1, \dots, x_n)$ for which in every region $|x_i| \leq \eta$, where η is any number such that $0 < \eta \leq h$, and $t > 0$, there is a subregion where $v > 0$. Suppose also that for v in the region (3) conditions (4), (5), and (6) are fulfilled and, for all values $t_0, x_{10}, \dots, x_{n0}$ such that $v(t_0, x_{10}, \dots, x_{n0}) \geq \alpha^2$, the mean $\varphi_0(t_0, x_{10}, \dots, x_{n0}) > \delta^2$, where α and $\delta \neq 0$. If these conditions are fulfilled, the equilibrium point is unstable under constantly acting perturbations.*

Proof. Choose the initial conditions x_{i0} for the solution $x_i = x_i(t)$ of system (1) so that $|x_{i0}| < \eta$ and $v(0, x_{10}, \dots, x_{n0}) \geq \alpha^2$, which is possible according to the conditions of the theorem. We shall consider the behavior

of the function v along this solution, assuming that the solution always remains in the region (3). Let us form the derivative of v by virtue of the equations of system (1):

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i + \mu \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i. \tag{7}$$

In the region (3), condition (4) is satisfied; therefore, integrating equality (7) along $x_i(t)$, we obtain, taking (4) into account, the inequality

$$v(t, x_1(t), \dots, x_n(t)) \geq$$

$$\geq v_0(0, x_{10}, \dots, x_{n0}) + \mu \int_0^t \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i \, dt = v_0 + \mu \int_0^t \varphi(x(t)) \, dt; \tag{8}$$

here $\varphi(x(t))$ has been introduced.

Let us divide $\int_0^t \varphi(x(t)) \, dt$ into integrals over intervals $t_{k+1} - t_k = \Delta t_k = l$ (the magnitude l will be chosen below) and add and subtract analogous integrals computed along the integral curves of system (2), $\bar{x}_{ik}(t)$, with initial values $\bar{x}_i(t_k) = x_i(t_k) = x_{ik}$, which $x_i(t)$ assumes at the points t_k:

$$\int_0^t \varphi(x(t))\,dt = \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \varphi(x(t))\,dt =$$

$$= \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \varphi(\bar{x}_k(t))\,dt + \int_{t_k}^{t_{k+1}} \left[\varphi(x(t)) - \varphi(\bar{x}_k(t))\right]\,dt. \tag{9}$$

We estimate the integrals along the solutions $\bar{x}_{ik}(t)$, using the positivity of the mean value φ_0: by the definition of the mean (4), there exists a function $\varkappa(t) \to 0$ as $t \to \infty$ such that

$$\int_{t_k}^{t_{k+1}} \varphi(\bar{x}_k(t))\,dt = (t_{k+1} - t_k)\left[\varphi_0(t_k, x_{1k}, \dots, x_{nk}) + \varkappa(\Delta t_k)\right]. \tag{10}$$

Averaging along the integral curves of the degenerate system was proposed in the works (^{5,6}) and was used by the author in the work (^{3}). Choose $l = t_{k+1} - t_k$ so large that

$$|\varkappa(\Delta t_k)| < \delta^2/4. \tag{11}$$

The boundedness of the perturbations R_i ($|R_i| \leq M$) and the Lipschitz condition for the right-hand sides f_i of system (2) make it possible to formulate Gronwall's inequality for the difference $x_i(t) - \bar{x}_{ik}(t)$ and to obtain on the interval $|\Delta t_k| \leq l$ the estimate

$$|x_i(t) - \bar{x}_{ik}(t)| \leq \mu M l e^{Nl}. \tag{12}$$

Using the smoothness conditions (6) and inequalities (12), one can indicate such a μ_0 that for $\mu < \mu_0$ the inequality

$$|\varphi(x(t)) - \varphi(\bar{x}_k(t))| < \delta^2/4, \tag{13}$$

will hold, and therefore

$$\int_{t_k}^{t_{k+1}} |\varphi(x(t)) - \varphi(\bar{x}_k(t))|\,dt < \frac{\delta^2}{4}\Delta t_k. \tag{14}$$

Thus, for

$$\int_{t_k}^{t_{k+1}} \varphi(x(t))\,dt,$$

by virtue of estimates (11) and (14) and condi

···of the theorem, $\varphi_0 \geqslant \delta^2$, it follows that

$$\int_{t_k}^{t_{k+1}} \varphi(x(t))\,dt \geqslant \Delta t_k \left[\delta^2 + \varkappa(t) - \frac{\delta^2}{4}\right] \geqslant \Delta t_k \frac{\delta^2}{2}. \tag{15}$$

At each step, in passing from t_{k+1} to t_{k+2}, there is an increase of v, so that the conditions of the theorem on the positivity of v and φ_0 are preserved as long as $x_i(t)$ does not leave the region (3); this makes it possible to apply successively inequalities (11), (14) and to obtain estimate (15) for any of the integrals entering the sum (9), so that for v along the solution $x_i(t)$ the inequality

$$v(t, x_1(t), \dots, x_n(t)) > v_0(0, x_{10}, \dots, x_{n0}) + \mu m l \delta^2/2 \tag{16}$$

will be valid.

If one assumes that $x_i(t)$ always remains in the region $|x_i| \leqslant h$, where conditions (6) are satisfied and the function v is bounded, then, letting t tend to infinity in inequality (8), and hence also the number of integrals m in inequality (16) tend to infinity, we arrive at a contradiction, since, according to inequality (16), in this case $v \to \infty$.

From the contradiction obtained it follows that the solution $x_i = x_i(t)$ at some instant of time leaves the region $|x_i| \leqslant h$, and since the initial conditions and the perturbations μR_i can be chosen arbitrarily small, the equilibrium point is unstable under permanently acting perturbations.

The theorem may be generalized by assuming that conditions (4) are fulfilled not in the whole h-neighborhood of the equilibrium point, and by proving a theorem of Chetaev type.

Following Chetaev, by the region $v > 0$ we shall mean some region of the neighborhood $|x_i| \leqslant h$ of the equilibrium point of system (2), bounded by the surface $v = 0$, in which the function v assumes positive values.

Theorem 2. *Suppose there exists a function $v(t, x_1, \dots, x_n)$ such that for $|x_i| \leqslant \eta$ and $t = 0$, where η is arbitrary, $0 < \eta \leqslant h$, there exists a region $v > 0$, and in it conditions (4), (5), and (6) are fulfilled; for all values $t_0, x_{10}, \dots, x_{n0}$ for which $v(t_0, x_{10}, \dots, x_{n0}) \geqslant \alpha^2$, the mean $\varphi_0(t_0, x_{10}, \dots, x_{n0}) \geqslant \delta^2$, where α and $\delta \neq 0$. Under these conditions the equilibrium point of system (2) is unstable under permanently acting perturbations.*

Proof of this system is constructed in the same way as the proof of Theorem 1. It is only necessary to take care that the solution $x_i(t)$ does not leave the region $v > 0$; otherwise condition (4) may be violated.

Theorem 3 extends the conclusions of Theorem 1 to the case when the inequality $v > 0$ entails $\varphi_0 > 0$ not at every point of region (3), but only in some part of it, which we shall call the region $\varphi_0 > 0$, while at the same time it is necessary to introduce a condition ensuring that the solution enters the region $\varphi_0 > 0$ on each time interval $[t_k, t_{k+1}]$.

Theorem 3. *Suppose the following conditions are satisfied:*

a) there exists a function $v(t, x_1, \dots, x_n)$ for which, in the region (3), conditions (4), (5), and (6) are satisfied;

b) in the region $\varphi_0 > 0$ the functions v and φ_0, for arbitrarily small x_i and any $t > 0$, assume positive values; moreover, for values $t_0, x_{10}, \dots, x_{n0}$ connected by the relation $v(t_0, x_{10}, \dots, x_{n0}) \geqslant \alpha^2$, the inequality $\varphi_0(t_0, x_{10}, \dots, x_{n0}) \geqslant 2\delta^2$ is satisfied, where α and $\delta \neq 0$;

c) one can specify a number l_1 such that on any piece of an integral curve of system (2), on a time interval l_1, lying entirely in region (3), there are points belonging to the region $\varphi_0 > 0$, at which $\varphi_0 \geqslant 2\delta^2$, if this integral curve starts from a point where $v \geqslant \alpha^2$.

Under these conditions the equilibrium point is unstable under permanently acting perturbations.

Proof. Consider an integral curve $x_i = x_i(t)$, issuing from a point x_{i0}, $1 \leqslant i \leqslant n$, arbitrarily close to the origin of coord–

coordinate and belonging to the region $\varphi_0 > 0$. Let at this point

$$v(0, x_{10}, \dots, x_{n0}) \geq a^2$$

and

$$\varphi_0(0, x_{10}, \dots, x_{n0}) \geq 2\delta^2.$$

By condition b), such a choice is possible. Keeping the scheme of the proof of Theorem 1, we shall define the lengths of the intervals $[t_k, t_{k+1}]$ differently. Choose l so large that inequality (11) is satisfied for $\Delta t_k \geq l$. Let now

$$l \leq t_{k+1} - t_k \leq l + l_1.$$

By choosing sufficiently small μ_1, we ensure that on any interval of length $l + l_1$, for $0 < \mu \leq \mu_1$, inequality (14) is satisfied.

According to condition c), on the integral curve $\bar{x}_{ik}(t)$ of system (2) there are points at which, for $t = t_{k+1}$, $\varphi_0 > 2\delta^2$. In view of conditions (6), φ_0 is continuous; therefore, imposing on μ one more restriction $0 < \mu \leq \mu_2$, we choose μ_2 so small that, by continuity of φ_0 and the smallness of $x_i - \bar{x}_{ik}$, on the curve $x_i(t)$ there are points at which $\varphi_0 > \delta$. For $0 < \mu \leq \mu_0$, where $\mu_0 = \min\{\mu_1, \mu_2\}$, on all intervals, starting with the first, inequalities (11), (14), and, consequently, (15) hold.

Supposing that the integral curve $x_i = x_i(t)$ remains in the region (3) and letting t tend to ∞, we arrive at a contradiction with conditions (6), since then v increases without bound. Thus the integral curve $x_i = x_i(t)$ necessarily leaves the region (3), and since the initial conditions x_{i0} and the perturbations μR_i can be taken arbitrarily small, the point of rest is unstable with respect to continually acting perturbations.

As an example, consider the well-known Mathieu equation with periodic coefficients

$$d^2x/dt^2 + \omega^2(1 - h\cos\nu t)x = 0.$$

Write it in the form of a system of equations

$$dx/dt = y, \qquad dy/dt = -\omega^2 x + \omega^2 hx\cos\nu t;$$

for small h, the term $h\omega^2 x\cos\nu t$ is regarded as a perturbation. The Lyapunov function is $\omega^2x^2 + y^2 = v$; its derivative is

$$dv/dt = 2\omega^2 hxy\cos\nu t.$$

The average of the expression $2\omega^2 hxy\cos\nu t$, computed along a solution of the unperturbed system with initial conditions $x(0) = x_0$, $y(0) = y_0$, is equal to $h\omega^2x_0y_0$ when $\nu = 2\omega$; this quantity is positive if x_0 and y_0 lie in the first and third coordinate angles. Thus, for small h, the system satisfies the conditions of Theorem 3, and this makes it possible to detect the principal parametric resonance $\nu = 2\omega$.

The theorems proved determine sufficient conditions for resonances in nonlinear oscillatory systems.

The author expresses his gratitude to V. M. Volosov for useful remarks.

Moscow State University
named after M. V. Lomonosov

Received
1 III 1967

REFERENCES

1. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Kharkov, 1892; 3rd ed., 1950.
2. N. G. Chetaev, DAN, 1, No. 9 (1934).
3. M. M. Khapaev, DAN, 176, No. 6 (1967).
4. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1963.
5. V. M. Volosov, DAN, 141, No. 1 (1961).

6. V. M. Volosov, DAN, 145, No. 5 (1962).

Note: Figure translations are in progress. See original paper for figures.