ON SOME BOUNDARY PROPERTIES OF GENERALIZED ANALYTIC FUNCTIONS

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Abstract

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ON SOME BOUNDARY PROPERTIES OF GENERALIZED ANALYTIC FUNCTIONS

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The present work is devoted to the study of the boundary behavior of generalized analytic functions (in the sense of I. N. Vekua (1)). The usual study of such behavior is carried out under considerable assumptions concerning the smoothness of the solutions under consideration. However, for a number of questions these assumptions are unnecessarily burdensome. For example, wishing to apply to generalized analytic functions the duality theory of extremal problems (see, for example, (2)), we encounter the necessity of investigating boundary properties under the most general assumptions, as is done in the theory of boundary properties of analytic functions (3).

In the present note we set forth a number of results concerning the transfer of certain important theorems of the theory of boundary values of analytic functions. We adhere to the basic definitions and notation of the book (1). At the same time, for the theorems formulated below we retain those names borne by the analogues of these theorems for ordinary analytic functions.

We shall consider the differential equation

$$\partial_{\bar z} w(z) + A(z)w(z) + B(z)\overline{w(z)} = 0, \tag{1}$$

where $A(z), B(z) \in \mathcal{L}_{p,2}(E)$, $p > 2$, E is the whole finite plane, $\partial_{\bar z} w = \frac{1}{2}(\partial w/\partial x + i\,\partial w/\partial y)$ (see (1), p. 149). Equation (1) with such coefficients has regular solutions. The class of such solutions in a domain G will be denoted by the symbol $U_{p,2}(A, B, G)$.

The equation

$$\partial_z w'(z) - A(z)w'(z) - \overline{B(z)}\,\overline{w'(z)} = 0 \tag{2}$$

is called conjugate to equation (1).

Let G be a finite domain with rectifiable boundary Γ, and let $\Omega_1(z,t)$, $\Omega_2(z,t)$ be the so-called basic kernels for the class $U_{p,2}(A, B, G)$ (1), p. 179). Let there be given on Γ a complex-valued function $F(t)$ of bounded variation. ($F(t)$ may be regarded as a function of the arc length s on the contour Γ, defined on the interval $[0, l]$, where l is the total length of Γ.) We shall call the expression

$$\frac{1}{2\pi i}\int_{\Gamma}\Omega_1(z,t)\,dF(t)-\Omega_2(z,t)\,d\overline{F}(t) \tag{3}$$

an integral of Cauchy-Stieltjes type.

This integral gives a regular solution of equation (1) for $z \notin \overline{\Gamma}$.

Let t_0 be some point of the line Γ, determined by the value s_0 of its arc. Denote by Γ_ε the part of the line Γ remaining after removing from Γ that arc whose endpoints are the points $t(s_0-\varepsilon)$, $t(s_0+\varepsilon)$ and which contains t_0.

The singular integral will be called the finite limit (if it exists) of the expression

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\Omega_1(t_0,t)\,dF(t)-\Omega_2(t_0,t)\,d\overline{F(t)},$$

writing in this case

$$\frac{1}{2\pi i}\int_{\Gamma}\Omega_1(t_0,t)\,dF(t)-\Omega_2(t_0,t)\,d\overline{F(t)}=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\Omega_1(t_0,t)\,dF(t)-\Omega_2(t_0,t)\,d\overline{F(t)}. \tag{4}$$

Consider the difference

$$\frac{1}{2\pi i}\left[\int_{\Gamma}\Omega_1(z,t)\,dF(t)-\Omega_2(z,t)\,d\overline{F(t)}-\int_{\Gamma_\varepsilon}\Omega_1(t_0,t)\,dF(t)-\Omega_2(t_0,t)\,d\overline{F(t)}\right]. \tag{5}$$

Draw the normal to Γ at the point t_0, and take z on the straight line $\overline{zt_0}$ making an angle $0 \le \psi_0 < \pi/2$ with the normal. The point z may lie either inside the domain G or outside it. We shall take the distance $\overline{zt_0}$ equal to ε.

Theorem 1 (the main lemma of Privalov; cf. (^{3}), p. 183). *If at the point t_0 there exists a finite derivative $F'(t_0)$, then expression (5) tends to the limit $+\frac{1}{2}F'(t_0)$ ($+\frac{1}{2}F'(t_0)$, if $z \to t_0$ from within G, and $-\frac{1}{2}F'(t_0)$, if $z \to t_0$ from outside G). The convergence to the limit is uniform with respect to ψ_0, $|\psi_0| \le \frac{1}{2}\pi\theta,\ \theta < 1$.*

Since $F'(t_0)$ exists almost everywhere on Γ, the assertion that (5) has the limit $\pm F'(t_0)$ holds for almost all points of Γ.

The proof is obtained with the aid of Privalov' s main lemma for ordinary analytic functions and estimates of the kernels $\Omega_1(z,t)$ and $\Omega_2(z,t)$, given by the formulas

$$\Omega_1(z,t) = \frac{1}{t-z} + O\left(|z-t|^{-2/p}\right), \qquad \Omega_2(z,t) = O\left(|z-t|^{-2/p}\right) \tag{6}$$

on p. 179 of the book (1).

Let us note some consequences.

Corollary 1. *If the singular integral (4) exists almost everywhere on* Γ*, then the Cauchy-Stieltjes type integral (3) has finite angular boundary values almost everywhere on* Γ*, as* $z \to t_0$*, equal to*

$$\frac{1}{2\pi i}\int_\Gamma \Omega_1(t_0,t)\,dF(t) - \Omega_2(t_0,t)\,d\overline{F(t)} \pm \frac{1}{2}F'(t_0); \tag{7}$$

conversely, if (3) has angular boundary values almost everywhere on Γ *(from inside* Γ *or from outside* Γ*), then the singular integral (4) exists almost everywhere on* Γ*.*

Let z and z^* be two points lying at distance ε from t_0, on one and the same straight line passing through t_0 at an angle ψ_0, $|\psi_0| < \pi/2$, to the normal and on different sides of Γ.

Corollary 2. *The difference of the values of the Cauchy-Stieltjes type integral inside and outside*

$$\frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)\,dF(t) - \Omega_2(z,t)\,d\overline{F(t)} - \int_\Gamma \Omega_1(z^*,t)\,dF(t) - \Omega_2(z^*,t)\,d\overline{F(t)}\right]$$

tends to the limit $F'(t_0)$*, when* ε *tends to zero, for all points* t_0 *of the line* Γ*, except, possibly, for points of a set of measure zero, independent of* ψ_0*, uniformly with respect to* ψ_0*,* $|\psi_0| \le \frac{1}{2}\pi\theta,\ \theta < 1$*.*

A particular case of Theorem 1 is the case of a Cauchy-Lebesgue type integral, when instead of $dF(t)$ in (3) one has $\varphi(t)\,dt$, where $\varphi(t)$ is a summable function on Γ.

Here we give only the formulation of Corollary 2 for this case.

Corollary 2′. The difference of the values of the Cauchy-Lebesgue type integral inside and outside Γ

$$\frac{1}{2\pi i}\left[\int_\Gamma \Omega_1(z,t)\varphi(t)\,dt - \Omega_2(z,t)\overline{\varphi(t)}\,d\bar{t} - \int_\Gamma \Omega_1(z^*,t)\varphi(t)\,dt - \Omega_2(z^*,t)\overline{\varphi(t)}\,d\bar{t}\right] \tag{8}$$

tends to the limit $\varphi(t)$ as ε tends to zero for every point t_0 of the line Γ, except possibly for points of a set of measure zero independent of ψ_0, uniformly with respect to ψ_0, $|\psi_0| \leq 1/2\pi\theta$, $\theta < 1$.

If the Cauchy-Stieltjes type integral (3) has almost everywhere on Γ angular boundary values $F'(t)$ as z tends to the points of Γ from inside G, then we shall call such an integral a Cauchy-Stieltjes integral. The Cauchy-Lebesgue integral is defined analogously.

Theorem 2 (Golubev-Privalov). Let $\varphi(t)$ be summable on Γ. The conditions

$$\frac{1}{2\pi i}\int_\Gamma \Omega_1(z^*,t)\varphi(t)\,dt - \Omega_2(z^*,t)\overline{\varphi(t)}\,d\bar{t} = 0, \qquad z^* \in \overline{G} \tag{9}$$

are necessary and sufficient for there to exist a function of the class $U_{p,2}(A,B,G)$, representable by a Cauchy integral, whose angular boundary values coincide almost everywhere with $\varphi(t)$.

Theorem 3. If a function of the class $U_{p,2}(A,B,G)$ is representable by the Cauchy-Stieltjes integral

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\,dF(t) - \Omega_2(z,t)\overline{dF(t)},$$

then it is representable by the Cauchy-Lebesgue integral of the function $F'(t)$, i.e.

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)F'(t)\,dt - \Omega_2(z,t)\overline{F'(t)}\,d\bar{t},$$

$$\lim_{z\to t} w(z) = F'(t)$$

almost everywhere (the limit in the sense of the angular boundary value).

Theorem 4. Let $\varphi(t)$ be a summable function on Γ and

$$\operatorname{Re}\left[\frac{1}{2i}\int_\Gamma w(t)\varphi(t)\,dt\right] = 0 \tag{10}$$

for all solutions of equation (2) continuous in $\overline{G}$. Then $\varphi(t)$ is the angular boundary value of some solution of equation (1).

Here the kernels Ω_1 and Ω_2 are assumed to be normalized with respect to the domain G (see (1), p. 193).

Next, there is a theorem which is an analogue of the Riesz brothers' theorem.

Theorem 5. Let Γ be an analytic contour. If

$$\mathrm{Re}\left[\frac{1}{2i}\int_\Gamma w(t)\,d\mu(t)\right] = 0 \tag{11}$$

for all solutions of equation (2) continuous in $\overline{G}$, where $\mu(t)$ is a function of bounded variation on Γ, then: a) $\mu(t)$ is absolutely continuous on Γ, and b) $d\mu(t) = \mu'(t)\,dt = w_1(t)\,dt$, where $w_1(t)$ is the boundary value of some solution of equation (1).

We note that in the proof of Theorem 5 the following fact is obtained incidentally:

The set of functions $\{\Omega_1(z,t), \Omega_2(z,t)\}$, as z varies outside Γ, is dense in the space $C(\Gamma)$ of functions continuous on Γ.

In conclusion, we note that it proves useful to consider classes of solutions of equation (1) consisting of functions with bounded mean moduli, analogous to the classes H_p and E_p studied in the theory of analytic functions. These questions will be considered in another note.

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Note: Figure translations are in progress. See original paper for figures.

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