GROUP PROPERTIES OF WAVE EQUATIONS FOR PARTICLES WITH ZERO MASS

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Abstract

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GROUP PROPERTIES OF WAVE EQUATIONS FOR PARTICLES WITH ZERO MASS

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The conformal invariance of the wave equations for particles with zero mass was established in works ([1]-[4]). But, as far as we know, nowhere has it been proved that the conformal group is the widest group admitted by these equations. In the present note we give the result of solving the problem of finding the widest local Lie group of point transformations for the homogeneous equations of Dirac, Maxwell, and a weak gravitational field.

We shall use only lower indices and shall sum over all repeated indices. The indices k, l, m, n take the values $1, 2, 3, 4$, and the indices p, q, r, t take the values $1, 2, 3$. By e_{pqr} and e_{klmn} we shall denote the usual permutation symbols in the spaces of three and four dimensions, respectively. The operators of the transformation group of the space $E_4(x)$ will be written in the form

$$X = \xi_k(x)\,\partial/\partial x_k, \tag{1}$$

and those of the space $E_{4+N}(x, u)$, where u_τ $(\tau = 1, \dots, N)$ are functions of x, in the form

$$\overline{X} = \xi_k(x, u)\frac{\partial}{\partial x_k} + \eta_\tau(x, u)\frac{\partial}{\partial u_\tau}. \tag{2}$$

The usual procedure for calculating the Lie group for systems of differential equations ([5]) shows that, for the equations considered here, the operator (2) has the special form

$$\overline{X} = X + s_{\tau\sigma}(x)u_\sigma\,\partial/\partial u_\tau, \tag{3}$$

and the coordinates ξ_k of the operator (1) satisfy the equations

$$\partial\xi_k/\partial x_l + \partial\xi_l/\partial x_k = \mu(x)\delta_{kl}, \tag{4}$$

which determine the 15-parameter group of conformal transformations of Minkowski space (^{6}). Solving system (4), we obtain the following basic operators X:

$$\begin{aligned} X_k &= \partial/\partial x_k, \\ X_{kl} &= x_l\partial/\partial x_k - x_k\partial/\partial x_l \quad (k < l), \\ T &= x_k\partial/\partial x_k, \\ Y_k &= (2x_{kx}l - |x|^2\delta_{kl})\partial/\partial x_l, \end{aligned} \tag{5}$$

where $|x|^2 = \sum_k x_k^2$. Introducing the matrix $S = \|s_{\sigma\tau}\|$ and noting that

$$\overline{X}u_\tau = s_{\tau\sigma}u_\sigma = (Su)_\tau,$$

we shall also write (3) in another form:

$$\overline{X} = X + S. \tag{6}$$

1. Dirac equations. The homogeneous Dirac equations have the form

$$\gamma_k\partial\psi/\partial x_k = 0, \tag{7}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

and γ_k are the four-row Dirac matrices. The widest Lie group of transformations admitted by equations (7) has order 17*. As a basis of the corresponding Lie algebra we may take the following operators, written in the form (6):

$$\overline{X} = X + \frac{1}{8}\frac{\partial\xi_k}{\partial x_l}(\gamma_k\gamma_l - \gamma_l\gamma_k - 3\delta_{kl}),$$

$$\overline{A} = \gamma_1\gamma_2\gamma_3\gamma_4, \tag{8}$$

$$\overline{B} = I,$$

where X runs through the system of operators (5), and I is the four-row identity matrix, giving the dilation operator of the function ψ. When the operator X

corresponds to the inhomogeneous Lorentz group, we obtain the known formula (7)

$$X = X + \frac{1}{8}\frac{\partial \xi_k}{\partial x_l}(\gamma_k\gamma_l - \gamma_l\gamma_k).$$

2. Maxwell equation . We shall use the four-dimensional notation for Maxwell' s equations

$$\partial F_{kl}/\partial x_m + \partial F_{lm}/\partial x_k + \partial F_{mk}/\partial x_l = 0,$$

$$\partial F_{kl}/\partial x_l = 0. \tag{9}$$

The widest Lie group for equations (9) also has order 17, and the corresponding basis operators, written in the form (3), have the form

$$\overline{X} = X - \sum_{k<l}\left(F_{km}\frac{\partial \xi_m}{\partial x_l} + F_{mk}\frac{\partial \xi_m}{\partial x_k}\right)\frac{\partial}{\partial F_{kl}}, \tag{10}$$

$$\overline{A} = \sum_{k<l}\widetilde{F}_{kl}\frac{\partial}{\partial F_{kl}},$$

$$\overline{B} = \sum_{k<l}F_{kl}\frac{\partial}{\partial F_{kl}}. \tag{11}$$

Here

$$\widetilde{F}_{kl} = \frac{i}{2}e_{klmn}F_{mn}$$

is the tensor dual to the electromagnetic-field tensor F_{kl}, and the operator X runs through the system of operators (5). Formula (10) shows that the quantities F_{kl} form a tensor with respect to conformal transformations, i.e., we have

$$F'_{kl} = F_{mn}\frac{\partial x_m}{\partial x'_k}\frac{\partial x_n}{\partial x'_l} \tag{12}$$

under conformal transformations $x'_k = x'_k(x)$. Indeed, passing in (12) to infinitesimal transformations, we obtain (10). When X corresponds to the inhomogeneous Lorentz group, the operators (10) have two inde-

* More precisely, we are dealing with an infinite-dimensional Lie algebra, which is connected with the linearity of equations (7). Namely, all transformations of

the form $\psi' = \psi + \varphi(x)$, where $\varphi(x)$ is any solution of (7), are admitted. We have excluded such transformations from consideration. The same remark is also valid for the other two equations.

** Yu. A. Danilov informed me that he too has considered the group properties of Maxwell' s equations.

dependent invariants

$$J_1 = \sum_{k<l} F_{kl}^2, \qquad J_2 = \sum_{k<l} F_{kl} \widetilde{F}_{kl}.$$

The full 17-parameter group has no invariants, i.e., it is transitive.

3. Equations of a weak gravitational field. The equations for a weak gravitational field have the form (8)

$$e_{pqr} \partial \Phi_{tr} / \partial x_q - \partial \Phi_{pt} / \partial x_4 = 0, \qquad \partial \Phi_{pq} / \partial x_q = 0, \tag{13}$$

where Φ_{pq} are the elements of a symmetric complex matrix with zero trace. As independent elements of this matrix we choose

$$u_1 = \Phi_{11}, \qquad u_2 = \Phi_{12}, \qquad u_3 = \Phi_{13}, \qquad u_4 = \Phi_{22}, \qquad u_5 = \Phi_{23}$$

and introduce the matrices

$$\beta_1 = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & -2 & 0 \end{Vmatrix}, \qquad \beta_2 = \begin{Vmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix},$$

$$\beta_3 = \begin{Vmatrix} 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{Vmatrix}.$$

Then we have the following basis operators of the largest Lie group for equations (13), written in the form (6):

$$
\begin{aligned}
\overline{X}_k &= \partial/\partial x_k, \\
\overline{X}_{12} &= x_2\partial/\partial x_1 - x_1\partial/\partial x_2 + \beta_3, \\
\overline{X}_{31} &= x_1\partial/\partial x_3 - x_3\partial/\partial x_1 + \beta_2, \\
\overline{X}_{23} &= x_3\partial/\partial x_2 - x_2\partial/\partial x_3 + \beta_1, \\
\overline{X}_{41} &= x_1\partial/\partial x_4 - x_4\partial/\partial x_1 + \beta_1, \\
\overline{X}_{42} &= x_2\partial/\partial x_4 - x_4\partial/\partial x_2 + \beta_2, \\
\overline{X}_{43} &= x_3\partial/\partial x_4 - x_4\partial/\partial x_3 + \beta_3, \qquad (14) \\
\overline{Y}_1 &= (2x_1x_k - |x|^2\delta_{1k})\partial/\partial x_k + 2(x_3\beta_2 - x_2\beta_3 + x_4\beta_1 - 3x_1I), \\
\overline{Y}_2 &= (2x_2x_k - |x|^2\delta_{2k})\partial/\partial x_k + 2(x_1\beta_3 - x_3\beta_1 + x_4\beta_2 - 3x_2I), \\
\overline{Y}_3 &= (2x_3x_k - |x|^2\delta_{3k})\partial/\partial x_k + 2(x_2\beta_1 - x_1\beta_2 + x_4\beta_3 - 3x_3I), \\
\overline{Y}_4 &= (2x_4x_k - |x|^2\delta_{4k})\partial/\partial x_k - 2(x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + 3x_4I), \\
\overline{T} &= x_k\partial/\partial x_k, \\
\overline{B} &= I.
\end{aligned}
$$

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