

# ON THE MULTIVALUEDNESS OF EXTREMAL SOLUTIONS IN PROBLEMS OF OPTIMAL CONTROL

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**Abstract**

**Full Text**

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**CYBERNETICS AND CONTROL THEORY**

**V. V. LARICHEVA, M. V. REIN**

**ON THE MULTIVALUEDNESS OF EXTREMAL SOLUTIONS IN PROBLEMS OF OPTIMAL CONTROL**

*(Presented by Academician A. A. Dorodnitsyn, 17 XII 1966)*

Consider an object whose motion is described by the system of equations

$$dx/dt = f(x, u), \quad x \in X, \quad u \in U, \quad (1)$$

where  $x = (x^1, \dots, x^n)$ ,  $f = (f^1, \dots, f^n)$ ,  $u = (u^1, \dots, u^r)$ , the functions  $f^j$  and  $\partial f^j / \partial x^i$  ( $j = 0, \dots, n$ ;  $i = 1, \dots, n$ ) are defined and continuous on the whole space  $X \times U$ . In accordance with (1), the problem of finding a control  $u(t)$  that gives a minimum to the functional

$$I = \int_{t_0}^{t_1} f_0(x, u) dt \quad (2)$$

in passing from the manifold  $S_1$  to the manifold  $S_2$  reduces to a boundary-value problem of the  $n$ -th order for system (1) together with the system

$$\frac{d\psi_i}{dt} = - \sum_{\alpha=0}^n \frac{\partial f^\alpha}{\partial x^i} \psi_\alpha, \quad (3)$$

where  $\psi_0 = \text{const} \leq 0$ , and the control is determined from the condition of maximum, with respect to  $u$ , of the function

$$\mathcal{H} = \sum_{\alpha=0}^n \psi_\alpha f^\alpha(x, u).$$

Let us introduce the concept of the relief of a functional. Consider a set  $M_1$ , which is an open domain in  $n$ -dimensional phase space, and a manifold  $S_2$ . Suppose that for some point  $x_0 \in M_1$  there exists an extremal (in particular,

optimal) solution transferring the phase point from the position  $x = x_0$  to the manifold  $S_2$ , with the value of the functional  $I = I_e(x_0)$ . Suppose that for every point  $x \in M_1$  one can construct an extremal solution in such a way that  $I_e$ ,  $\partial I_e / \partial x$ , and  $\psi_i$  will be continuous functions of the points of the set  $M_1$ , and that  $I_e = I_e(x_0)$  at the point  $x = x_0$ . Then the hypersurface  $I = I_e(x)$ , where  $x \in M_1$ , will be called the relief (corresponding to the given extremal solution) of the functional over the set  $M_1$ . Hypersurfaces of equal level of the functional  $I_e(x) = \text{const}$  on the set  $M_1$  will be called characteristics of the relief.

We have defined the relief over a set of initial points. Analogously, one introduces a relief over a set  $M_2$  of terminal points (for a known extremal solution transferring the phase point from the manifold  $S_1$  to some point  $x_1 \in M_2$ ).

Let one of the manifolds, for example  $S_2 \subset M_2$ , be closed and bounded. Then the following holds.

**Theorem 1.** *Every relief of the functional over the set  $M_2$ , corresponding to an extremal solution of the transfer of the phase point from the mani-*

*mapping  $S'_1$  into the point  $x_1 \in M_2$ , gives rise to at least two extremal solutions of the problem of transfer from the manifold  $S'_1$  to the manifold  $S_2 \subset M_2$ .*

The set  $M_2$  is open, and  $\text{grad } I_e$  is defined at every point  $x \in M_2$ . If  $\text{grad } I_e(x) \neq 0$ , then through the point  $x$  one can draw a hypersurface  $S$  on which  $\min I_e$  is attained at the point  $x$ . Then at the point  $x$ , on the one hand, the conditions

$$\partial S / \partial x^i = \lambda \partial I_e / \partial x^i, \quad i = 1, \dots, n, \quad \lambda = \text{const}, \quad (4)$$

will be satisfied, and, on the other hand, the transversality conditions (1)

$$\psi_i = k \partial S / \partial x^i, \quad i = 1, \dots, n, \quad k = \text{const}, \quad (5)$$

whence it follows that if  $\text{grad } I_e(x) \neq 0$ , then the vector  $\psi$  determines the direction of the normal to the characteristic at the point  $x \in M_2$ . At points where  $\text{grad } I_e = 0$ , we have  $\psi_i = 0$  by virtue of (4), (5) and the continuity of  $\psi_i$ . Consequently, the solutions corresponding to  $\min I_e$  and  $\max I_e$  on  $S_2$  satisfy the transversality conditions in the problem of transfer from  $S'_1$  to  $S_2$ . ( $\min I_e$  and  $\max I_e$  are attained on  $S_2$  because  $S_2$  is closed and bounded.)

**Corollary of Theorem 1.** *If the manifold  $S_1 \subset M_1$  is also closed and bounded and if there exists an extremal solution transferring the phase point from the position  $x_0 \in M_1$  to the position  $x_1 \in M_2$ , then it gives rise to at least four extremal solutions transferring the phase point from the manifold  $S_1 \subset M_1$  to the manifold  $S_2 \subset M_2$ .*

Indeed, the known extremal solution from the point  $x_0$  to the point  $x_1$  gives rise to a relief over the set  $M_2$ , consisting of extremal solutions transferring the phase point from  $x_0$  to an arbitrary point of the set  $M_2$ . By Theorem 1, to

this relief there correspond two extremal solutions transferring the phase point from the position  $x_0$  to the manifold  $S_2$ . Each of these two solutions gives rise to its own relief over the set  $M_1$ . On each relief, by Theorem 1, there exist at least two (and therefore four on both reliefs) extremal solutions transferring the phase point from the manifold  $S_1$  to the manifold  $S_2$ .

If the manifolds  $S_1, S_2$  are hypersurfaces of dimension  $n - 1$ , then among the four extremal solutions it is not difficult to single out the best one in the sense of  $\min I$ , since three extremal solutions will almost always intersect the hypersurfaces  $S_1, S_2$ .

Theorem 1 and its corollary are important in the case when the dimension of the manifolds  $S_1, S_2$  (or of one of them) is less than  $n - 1$ . Here there is no additional criterion distinguishing the best extremal solution from the others.

In applications, as a rule, there is no possibility of verifying the fulfillment of sufficient optimality conditions for the solution of a variational problem. Therefore Theorem 1 (and the corollary) allow one to hope for the acquisition of new extremal solutions better than those known in the applied literature.

As an example, let us briefly consider the problem of the optimal (in the sense of minimizing the integral of the square of the control acceleration\*) transfer of a material point in a central gravitational field from one elliptic orbit to another in a given time  $t_1 - t_0$ . We shall use the variables (3)  $p, x, y$ , where  $p$  is the focal parameter, and  $x, y$  are related to the eccentricity  $e$ , the angular distance of the pericenter  $\omega$ , and the polar angle  $\varphi$  by the relations  $x = e \cos(\varphi - \omega)$ ,  $y = e \sin(\varphi - \omega)$ .

In the space of the variables  $p, x, y$ , the motion of the point under consideration admits a simple geometric interpretation. Indeed, transfer from one circular orbit to another corresponds in the space  $p, x, y$  to transfer from the point  $(p_0, 0, 0)$  to the point  $(p_1, 0, 0)$ . Transfer from one elliptic orbit to another corresponds in the space  $p, x, y$  to transfer from a circle of radius  $e_0$  in the plane  $p = p_0$  to a circle of radius  $e_1$  in

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\* On such functionals and the corresponding literature, see, for example, (2).

plane  $p = p_1$ . Applying the corollary of Theorem 1, we conclude that in the problem under consideration, in passing from an elliptic orbit to an elliptic one there will be at least 4 times as many extremal solutions as in passing from a circular orbit to a circular one.

On the basis of Theorem 1 (and the corollary), one can draw a practical conclusion for the numerical solution of optimal-control problems on an electronic computer. Suppose that the solution transferring the phase point from the position  $x_0 \in S_1$  to the position  $x_1 \in S_2$  is an extremal solution of the problem of optimal transfer from the manifold  $S_1$  to the manifold  $S_2$ . With the aid of a computer it is easy to find extremal solutions from a position  $x'_0 \in S_1$ , close to

$x_0 \in S_1$ , to the manifold  $S_2$ , and from the manifold  $S_1$  to a position  $x'_1 \in S_2$ , close to  $x_1 \in S_2$ . If the functional on one of the latter two solutions turns out to be smaller than on the extremal under investigation, then, obviously, there exists a better solution of the problem of transfer from  $S_1$  to  $S_2$ .

In [4] we have investigated the nonuniqueness of extremal solutions and given a method for the practical determination of extremals.

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*Note: Figure translations are in progress. See original paper for figures.*

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