AN APPROXIMATE MINIMAX PROPERTY OF THE TEST \(R^2\)

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Abstract

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MATHEMATICS

LI HOANG TU

AN APPROXIMATE MINIMAX PROPERTY OF THE TEST R^2

(Presented by Academician Yu. V. Linnik on 3 I 1968)

Let $X_1, \dots, X_N$ be independent p-dimensional vectors obeying the normal law $N(\xi, \Sigma)$ (here, as usual, $\xi = EX_i$, $\Sigma = E(X_i - \xi)(X_i - \xi)^T$ is the vector of means and the correlation matrix; vectors are written as columns, T denotes transposition). Put

$$N\overline{X} = \sum_{i=1}^{N} X_i, \qquad S = \sum_{i=1}^{N} (X_i - \overline{X})(X_i - \overline{X})^T,$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

(where Σ_{22}, S_{22} are $(p-1) \times (p-1)$-matrices).

$$\rho^2 = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}/\Sigma_{11}.$$

We shall test the statistical hypothesis $H_0 : \rho^2 = 0$ against the alternative $H_1 : \rho^2 = \delta,\ 0 < \delta < 1$, and against the alternative $H_1' : \rho^2 \geq \delta$, with level $\alpha,\ 0 < \alpha < 1$.

To solve the two indicated problems one usually uses the R^2-test, which rejects the hypothesis H_0 when $R^2 = S_{12}S_{22}^{-1}S_{21}/S_{11} > C$, where the constant C is determined by the level α.

In the case $N \leq p$, if ξ is unknown, or $N \leq p - 1$, if ξ is known, the test R^2 is equal to a constant (if it exists).

In 1964, H. Giri and J. Kiefer proved the minimax property of the test R^2 for the case $p = 3,\ N = 4$ (or $N = 3$, when ξ is known).

The study of samples of size $N > 4$ for $p = 3$ encounters considerable difficulties. However, for $N \to \infty$, as in ([3]), we proved the ε-minimaxity of the test R^2 for arbitrary values $p \geq 3$ ($\varepsilon = O(1/N^{1-\varepsilon_1})$, where ε_1 is any prescribed small positive number).

In the present paper we apply the methods used by Yu. V. Linnik, Yu. V. Prokhorov, O. V. Shalaevskii, H. Giri, and J. Kiefer in [3,4,1], in order to prove ε-minimaxity for the test R^2 when $p = 3$, N is large. ($\varepsilon = O(1/N^k)$, where k is any prescribed natural number.)

Theorem 1. *The test R^2 : $R^2 > C$ for testing the hypothesis H_0 against H_1 is ε-minimax for any level $\alpha \in (0,1)$.*

For any $\varepsilon > 0$ the relation holds:

$$\mathrm{Sup}_\Phi \inf_{\theta\in H_1} E_\theta\Phi - \inf_{\theta\in H_1} E_\theta\Phi_N \leq \varepsilon \tag{1}$$

for $N > N_0(\varepsilon)$. Here $\theta = (\xi, \Sigma)$ is the parameter; Φ_N is the test R^2; Φ ranges over all tests of level $\leq \alpha$.

More precisely: if the level $\alpha = \alpha_N$ is subject to the conditions

$$O(\lambda/N^k) \leq \alpha \leq \lambda - O(1/\ln N) \tag{2}$$

and

$$1/N \ln N \leq \delta \leq 1, \tag{3}$$

then

$$\mathrm{Sup}_\Phi \inf_{\theta\in H_1} E_\theta\Phi - \inf_{\theta\in H_1} E_\theta\Phi_N = O(1/N^k). \tag{4}$$

When the hypothesis H_1 is replaced by the hypothesis H_1', an analogous theorem holds:

Theorem 2. *The test R^2: $R^2 > C$ for testing H_0 against the composite hypothesis H_1' is ε-minimax for any $\alpha \in (0,1)$ and $0 < \delta < 1$. Under the conditions of Theorem 1, relation (1) holds with H_1 replaced by H_1'.*

To prove these theorems we establish a lemma asserting the existence of an approximate solution λ of the Giri–Kiefer integral equation.

This equation has the form (the case $p = 3$)

$$\int_\Gamma \left[1+\sum_{i=2}^{3} r_i\left(\frac{1-\delta}{\gamma_i}-1\right)\right]^{-N/2} \sum_{\beta_2=0}^{\infty}\sum_{\beta_3=0}^{\infty} \frac{\Gamma(N/2+\beta_2+\beta_3)}{\Gamma(N/2)}\times$$
$$\times \prod_{i=2}^{3}\left\{\frac{\Gamma((N-i+2)/2+\beta_i)}{\Gamma((N-i+2)/2)(2\beta_i)!}\left(\frac{4r_i a_i^2}{1+\sum_{j=2}^{3} r_j\left(\frac{1-\delta}{\gamma_j}-1\right)}\right)^{\beta_i}\right\} d\lambda(\Delta) =$$
$$= F\left(\frac{N}{2},\frac{N}{2},1,C\delta\right),$$

where Γ is the simplex (δ_2,δ_3), $\delta_i \geq 0$, $\delta_2+\delta_3=\delta$; (r_2,r_3) are parameters, $r_2, r_3 > 0$, $r_2+r_3=C$; $\gamma_i = 1-\sum_{j=2}^{i}\delta_j$, $\gamma_1=1$, $a_i^2=\delta_i\gamma_3/\gamma_{i-1}\gamma_i$; F is the hypergeometric function.

Lemma. *As* $N\to\infty$ *and* $1/N\ln N \leq \delta \leq 12K\ln n/N$, $2/\ln N \leq C \leq 2K\ln n/N$, *there exists a continuous probability distribution* $\lambda_{C\delta N}$ *on* Γ *which, when substituted into (2), gives a residual, uniformly in* $r_2, r_3 \geq 0$, $r_2+r_3=C$, *not exceeding* $O(1/N^k)$, *where* k *is any fixed prescribed natural number.*

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.