

**ON  
\\(G\\)-COMPOSABILITY  
AND \\(G\\)-  
COMPLEMENTABILITY**

MATHEMATICS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON $G$ -COMPOSABILITY AND $G$ -COMPLEMENTABILITY**

*(Presented by Academician P. S. Novikov on 10 V 1967)*

In this note only compact sets that are closures of open sets will be considered; for brevity we shall call them **bodies**.

Let a group  $G$  act in a topological space  $R$ , possessing the following property: if  $A$  and  $B$  are bodies, then in the group  $G$  there exists a transformation  $g$  such that  $gA \subset B$ .

If the intersections  $P_i \cap P_j$ ,  $a_i P^i \cap a_j P^j$  ( $i \neq j$ ), where  $a_i \in G$ , are either empty or consist of points that are boundary points for the intersecting bodies, and

$$C = \bigcup_{i=1}^n P_i, \quad C' = \bigcup_{i=1}^n a_i P^i,$$

then  $C$  and  $C'$  will be called  **$G$ -composable**, and we shall write

$$C \sim_g C',$$

or simply  $C \sim C'$ .

If for bodies  $A$  and  $A'$  there can be found such bodies  $P_i$  and such transformations  $a_i \in G$  ( $i = 1, \dots, n$ ) that the intersections  $P_i \cap P_j$ ,  $a_i P^i \cap a_j P^j$  ( $i \neq j$ ),  $A \cap P_i$ ,  $A' \cap a_i P^i$  ( $i = 1, \dots, n$ ), are either empty or consist of boundary points, and

$$A \cup \bigcup_{i=1}^n P_i = A' \cup \bigcup_{i=1}^n a_i P^i,$$

then  $A$  and  $A'$  will be called  **$G$ -complementable**.

**Theorem 1.** Two  $G$ -complementable sets  $A$  and  $A'$  are always  $G$ -composable.

**Proof.** Given

$$A \cup P = A' \cup P',$$

where

$$P = \bigcup_{i=1}^n P_i, \quad P' = \bigcup_{i=1}^n a_i P^i, \quad a_i \in G \quad (i = 1, \dots, n).$$

If  $A = A'$ , there is nothing to prove; hence one may assume that one of them is not a part of the other; let, for example,  $A' \not\subset A$ , then  $\overline{A' \setminus A}$  is a body.

By the property of the group  $G$  formulated above, there exists a transformation  $\alpha \in G$  such that  $\alpha P \subset A$ , and such an  $\alpha' \in G$  that

$$\alpha' P' \subset \overline{A' \setminus A}.$$

Perform this transformation  $\alpha$  on  $P$ , and remove those parts of  $A$  which have been covered by the body  $\alpha P$ ; denote them by  $P^*$ , and send them by the transformation  $\alpha^{-1}$  to the place where the body  $P$  was previously located.

We have:

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^* \sim A,$$

since the body

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^*$$

is obtained from  $A$  by rearranging parts. In the same way we perform the transformation  $\alpha'$  on  $P'$ , and remove those parts of  $A'$  which have become covered by the body  $\alpha' P'$ ; denote them by  $P'^*$ , and by the transformation  $(\alpha')^{-1}$  send them to the places freed by the removal of  $P'$ .

We have:

$$(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^* \sim A',$$

since the body

$$(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^*$$

is obtained from  $A'$  by rearranging parts.

Introduce the new bodies

$$Q = A \cup P = A' \cup P'$$

and

$$R = \overline{Q \setminus \alpha P \setminus \alpha' P'}.$$

Since  $\alpha P \cap \alpha' P'$  is empty, the bodies

$$R \cup \alpha P$$

and

$$R \cup \alpha' P'$$

will be  $G$ -composable.

We have the equalities

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^* = R \cup \alpha' P', \quad (\overline{A' \setminus \alpha' P'}) \cup$$

$$\frac{\cup (\alpha')^{-1} P'^*}{(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^*} = \overline{R} \cup \alpha P,$$

but earlier we had  $(\overline{A \setminus aP}) \cup a^{-1}P^* \sim A$ ,  $(A' \setminus a'P') \cup (a')^{-1}P^* \sim A'$ , and, by the transitivity of  $G$ -composability,  $A \sim A'$ . The theorem is proved.

Let us derive from this theorem the following:

**Theorem 2.** *In three-dimensional Euclidean space, any two polyhedra can be decomposed into a finite number of pairwise similar polyhedra.*

Consider two equal tetrahedra  $T$  and  $T'$ . It is known that the tetrahedron  $T$  can be decomposed into two similar-to- $T$  tetrahedra  $T_1$  and  $T_2$ , and into two prisms  $\Pi_1$  and  $\Pi_2$ . The tetrahedron  $T'$  can likewise be decomposed into three tetrahedra similar to it and into four prisms:  $T' \sim T'_1 \cup T'_2 \cup T'_3 \cup \Pi'_1 \cup \Pi'_2 \cup \Pi'_3 \cup \Pi'_4$ .  
 a)  $T \sim T_1 \cup T_2 \cup \Pi_1 \cup \Pi_2$ . But it is known that there exists a prism  $\Pi'_5$  such that  $\Pi'_5 \sim \Pi'_1 \cup \Pi'_2 \cup \Pi'_3 \cup \Pi'_4$ ; then, evidently: b)  $T' \sim T'_1 \cup T'_2 \cup T'_3 \cup \Pi'_5$ .

Applying our theorem to the equalities a) and b) (taking into account that  $T = T'$ ,  $T_1$  is similar to  $T'_1$ ,  $T_2$  is similar to  $T'_2$ , and  $\Pi$  and  $\Pi'_5$  can be decomposed into pairwise similar ones), we obtain

$$\overline{T \setminus T_1 \setminus T_2 \setminus \Pi_1} \sim \Pi_2, \quad \overline{T' \setminus T'_1 \setminus T'_2 \setminus \Pi'_5} \sim T'_3.$$

The prism  $\Pi_2$  and the tetrahedron  $T'_3$  are similarly composed by virtue of our theorem. Thus, any tetrahedron is similarly composed with a prism. It follows from this that any two three-dimensional polyhedra can be decomposed into a finite number of pairwise similar polyhedra.

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*Note: Figure translations are in progress. See original paper for figures.*

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