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Abstract

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CALCULATION OF THE WAVE NUMBERS OF NORMAL WAVES IN SPHERICALLY LAYERED ANISOTROPIC MEDIA BY THE IMPEDANCE RECALCULATION METHOD*

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1. The angular wave numbers $\nu_j = \alpha_j + i\beta_j$ of normal waves traveling in the angle θ (r, θ, φ are spherical coordinates) in a medium with dielectric-permittivity tensor $\varepsilon(r)$ are determined as the roots of the impedance equation $\mathfrak{Z}$ (see (8) (1)), which transforms the amplitudes of fields normal to the r-axis:

$$\begin{pmatrix} E_\theta \\ E_\varphi \end{pmatrix} = \mathfrak{Z} \begin{pmatrix} H_\theta \\ H_\varphi \end{pmatrix}; \qquad \mathfrak{Z} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}. \tag{1}$$

Let us write this equation in the form

$$\det |\mathfrak{Z}(\bar{r}, 0; \nu) - \mathfrak{Z}(\bar{r}, \infty; \nu)| = 0. \tag{2}$$

Here $\mathfrak{Z}(\bar{r}, 0; \nu)$ is the impedance recalculated from the point $r = 0$, where $\mathfrak{Z} = \mathfrak{Z}_0$, to the point $r = \bar{r}$, chosen arbitrarily in the interval $(0, \infty)$, and $\mathfrak{Z}(\bar{r}, \infty; \nu)$ is recalculated from the point $r = \infty$, where $\mathfrak{Z} = \mathfrak{Z}_\infty$, to the point $\bar{r}$. $\mathfrak{Z}_0$ corresponds to the boundedness conditions for the fields as $r \to 0$, and $\mathfrak{Z}_\infty$ to the radiation conditions as $r \to \infty$. In (1–3), the recalculation of $\mathfrak{Z}_\infty$ was replaced by two recalculations from $r = \infty$ to $r = \bar{r}$ of the impedances of waves Z_y, Z_z, and X of the ordinary (o) and extraordinary (e) types. From these, by formulas (12) (1), $\mathfrak{Z}(\bar{r}, \infty; \nu)$ was constructed. These recalculations were performed by numerical integration on a computer of equations (9) (1) from r_∞, replacing the point $r = \infty$, to $r = \bar{r}$ of the impedance functions u and χ, which determine Z_y, Z_z, and X.

In (2,3), the recalculation of $\mathfrak{Z}_0$ was performed by analytic formulas in which asymptotic approximations of Hankel functions (4,5) were used, leading to uncontrolled errors (we shall call this method semianalytic). In (6) a method was given in which all wave impedances are obtained by integration on a computer. In principle it provides high accuracy, but it requires the specification of exact initial conditions, since the integration of the impedance functions of the extraordinary wave is unstable with respect to small perturbations of the initial conditions. Here we develop a method for constructing (2) based on direct integration of the matrix impedance operator $\mathfrak{Z}$ through the corresponding "sweep" equation. According to (7), owing to the continuity of the operator transforming $\mathfrak{Z}_{r_1}$ into $\mathfrak{Z}(r_2, r_1; \nu)$, where r_1 and r_2 are arbitrary points in $(0, \infty)$, such an integration is stable with respect to small perturbations of $\mathfrak{Z}_{r_1}$.

2. To derive the sweep equation for $\mathfrak{Z}$, write the normal wave as

$$\mathbf{e}_j(r) L_\nu^{(1,2)}(\theta) \exp(-i\omega t), \qquad j = 0, 1, 3, \tag{3}$$

where $\mathbf{e}(r)$ is a 4-vector with components $rE_\theta, rE_\varphi, rH_\theta$, and rH_φ, and $L_\nu^{(1,2)}(\theta)$ is the phase factor of the wave, having the asymptotic form $\approx (\sin\theta)^{-1/2} \exp(i\nu_j \theta)$. From Maxwell' s equations in spherical coordinates we obtain, neglecting the differentiation of $\sin\theta$ with respect to θ, the system of equations

$$\varepsilon_{rr} \frac{d\mathbf{e}}{dr} = ik\mathcal{A}\mathbf{e}; \qquad \mathcal{A} = \begin{pmatrix} -S\varepsilon_{r\theta} & -S\varepsilon_{r\varphi} & 0 & \varepsilon_{rr} - S^2 \\ 0 & 0 & -\varepsilon_{rr} & 0 \\ -\tilde{\varepsilon}^* & \varepsilon_{rr} S^2 - \delta & 0 & S\varepsilon_{\varphi r} \\ \Delta & -\varepsilon^* & 0 & -S\varepsilon_{\theta r} \end{pmatrix}, \tag{4}$$

where

$$S = \nu/kr; \quad \Delta = \varepsilon_{\theta\theta}\varepsilon_{rr} - \varepsilon_{\theta r}\varepsilon_{r\theta}; \quad \varepsilon^* = \varepsilon_{r\varphi}\varepsilon_{\theta r} - \varepsilon_{\theta\varphi}\varepsilon_{rr}; \quad \delta = \varepsilon_{\varphi\varphi}\varepsilon_{rr} - \varepsilon_{\varphi r}\varepsilon_{r\varphi};$$

$$k = \omega/c_0; \quad c_0 = 3 \cdot 10^{10}\ \text{cm/sec}, \quad \tilde{\varepsilon}^* = -\varepsilon_{\varphi r}\varepsilon_{r\theta} + \varepsilon_{\varphi\theta}\varepsilon_{rr}.$$

In the terminology of work (7), $\mathcal{A}$ is the Breizman matrix. The eigenvectors $\mathbf{e}_j(r)$ satisfy equation (4) under the boundary conditions of boundedness of the fields as $r \to 0$ and radiation as $r \to \infty$, which is possible for discrete values of the parameter $\nu = \nu_j$, called eigenvalues. Dividing the matrix $\mathcal{A}$ into 2×2 blocks $\mathcal{A}_{m,p}$ $(m, p = 1, 2)$ and taking (1) into account, we obtain the desired propagation equation

* The method of impedance recalculation for a scalar $\mathfrak{Z}$ was used as early as the 1930s in the theory of long lines. Here $\mathfrak{Z}$ is a matrix, and in (7) it is a functional operator. The method of sweeping boundary conditions is its generalization used in computational mathematics.

$$-\frac{i\varepsilon_{rr}}{k}\frac{d}{dr}\mathfrak{Z} = \mathcal{A}_{12} + \mathcal{A}_{11}\mathfrak{Z} - \mathfrak{Z}\mathcal{A}_{22} - \mathfrak{Z}\mathcal{A}_{21}\mathfrak{Z}, \tag{5}$$

An analogous equation for a plane-layered medium was obtained by Budden (^{8}). For the numerical construction of (2), $\mathfrak{Z}$ is marched on a computer through equation (5) for the initial values $\mathfrak{Z}_0$ at $r = 0$ and $\mathfrak{Z}_\infty$ at $r = r_\infty$ to the intermediate point $r = \bar{r}$. The requirement of continuity of the fields $E_\theta, E_\varphi, H_\theta$, and H_φ at this point is expressed by a system of two linear equations:

$$\mathfrak{Z}(\bar{r}, 0; \nu)\begin{pmatrix} H_\theta \\ H_\varphi \end{pmatrix} = \mathfrak{Z}(\bar{r}, \infty; \nu)\begin{pmatrix} H_\theta \\ H_\varphi \end{pmatrix},$$

whose compatibility condition is equation (2).

3. To compute $\mathfrak{Z}_\infty$, we construct $\mathbf{e}(r)$ in the asymptotic approximation as $r \to \infty$. We seek a solution of equation (4) as $r \to \infty$ in the form $\mathbf{e} = \underline{\mathbf{e}}\exp\int u\,dr$. For $\underline{\mathbf{e}}$ we obtain the equation $(ik\mathcal{A} - \varepsilon_{rr}u\mathcal{E})\underline{\mathbf{e}} = 0$, whence follows the compatibility equation $\det|ik\mathcal{A} - \varepsilon_{rr}u\mathcal{E}| = 0$, determining the wave numbers u. In the notation (1) it has the form:

$$u^4 + au^3 + (e + b - cf)u^2 + (ea - cg - bf)u + (eb - bg) = 0. \tag{6}$$

Equation (6) has four roots $u_\pm^{e,o}$, corresponding to the extraordinary (e) wave and the ordinary (o) wave, running away from the center ($+$) and toward the center ($-$). To each wave one can assign three impedances $Z_y = E_\theta/H_\varphi$, $Z_z = H_\theta/E_\varphi$, and $X = E_\varphi/H_\varphi$, determined from the equation $(ik\mathcal{A} - \varepsilon_{rr}u_\pm^{e,o}\mathcal{E})\mathbf{e} = 0$. Only waves receding from the center, i.e. $u_+^{e,o}$, should be included in the solution $\mathbf{e}(r)$ $(r \to \infty)$. Omitting the index $+$, we obtain for the amplitude of waves of type e and o

$$\underline{\mathbf{e}}^{e,o} = C^{e,o}\begin{pmatrix} Z_y^{e,o} \\ X^{e,o} \\ Z_z^{e,o}X^{e,o} \\ 1 \end{pmatrix}; \qquad Z_y^{e,o} = \frac{1}{\Delta}\left[\frac{\varepsilon_{rr}}{ik}u^{e,o} + \varepsilon^* X^{e,o} \pm \varepsilon_{\theta r}S\right]; \tag{7}$$

$$Z_z^{e,o} = -\frac{u^{e,o}}{ik}; \qquad X^{e,o} = -\frac{g + fu^{e,o}}{e + (u^{e,o})^2},$$

where C^e and C^o are arbitrary constants. We seek the solution $\mathbf{e}(r)$ $(r \to \infty)$ in the form

$$\mathbf{e}(r) = C^e\underline{\mathbf{e}}^e\exp\int u^e\,dr + C^o\underline{\mathbf{e}}^o\exp\int u^o\,dr; \tag{8}$$

it satisfies the radiation conditions as $r \to \infty$. Eliminating $C^{e,o} \exp \int u^{e,o} dr$ from the system of equations (8), we obtain equations relating E_θ, E_φ to H_θ, H_φ by the matrix $\mathfrak{Z}_\infty$ with elements

$$z_{11} = (Z_y^e - Z_y^o)/\delta_0; \qquad z_{12} = (X^e Z_z^e Z_y^o - X^o Z_z^o Z_y^e)/\delta_0; \tag{9}$$

$$z_{21} = (X^e - X^o)/\delta_0; \qquad z_{22} = (Z_z^e - Z_z^o) X^e X^o/\delta_0; \qquad \delta_0 = X^e Z_z^e - X^o Z_z^o.$$

(In (^{1}), an error was made in formula (12) for z_{22}.)

4. In computing $\mathfrak{Z}_0$ we shall restrict ourselves to the practically important case when, in the interval $(0, \bar{r})$, ε is a scalar function of r, and one may set $z_{11} = z_{22} = 0$, choosing the corresponding wave polarizations. In this case the system

of equations (5) splits into 2 equations. Owing to the boundedness of the fields at $r = 0$, $z_{12}(0) = \infty$, and $z_{21}(0) = 0$. The first initial condition is inconvenient for shooting; therefore we replace z_{12} by the function $v = [i\varepsilon k z_{12}]^{-1}$, and replace z_{21} by the function $w = -z_{21} i/k$. For them we obtain the equations

$$v_r' - \varepsilon(\varepsilon^{-1})_r' v - k^2(\varepsilon - S^2)v^2 - 1 = 0; \qquad w_r' - k^2(\varepsilon - S^2)w^2 - 1 = 0. \tag{10}$$

$Z(\bar{r}, 0; v)$ is obtained by shooting through (10) with the conditions $v(0) = 0$ and $w(0) = 0$.

Fig. 1. Solid line—$u(h)$ for $a = 6370$ km; dashed line—$u(h)$ for $a = \infty$

5. The roots of (2) are computed on an electronic computer by the method of successive approximations, for example by Newton' s method, when the correction $\Delta v_j^{(n)}$ to $v_j^{(n)}$ at the n-th step is equal to $-F(v_j^{(n)})/(dF/dv)$, where $F(v)$ is the left-hand side of (2). dF/dv is computed as the ratio of finite differences $\Delta F/\Delta v$.

6. As the simplest numerical example, let us consider a medium characterized by the following scalar function $\varepsilon(r)$: $|\varepsilon| = \infty$ for $r < a$; $\varepsilon = 1$ for $a < r < c$, and for $r > c = a + 51$

$$\varepsilon(r) = V/(1 + is); \qquad V = \omega_0^2/\omega^2; \qquad s = \nu_{eff}/\omega; \qquad \omega_0^2 = 4\pi N_e e^2/m, \tag{11}$$

where

$$N_e = \begin{cases} 6.73(h-51), & 51<h<65; \\ 62.8\exp[0.3(h-65)]+3.14, & h>65; \end{cases} \qquad \nu_{eff} = 5{\cdot}10^5\exp[-0.148(h{-}89)]; \tag{11'}$$

$h = r - a$ is the height in kilometers above the Earth. Case (11) refers to the propagation of longwave radio waves in the “Earth—lower ionosphere” waveguide under summer noon conditions at middle latitudes over the sea. For the scalar function $\varepsilon(r)$, equation (4), and hence also (5), are exact, and (5) splits into two equations, for waves of type TH_j and TE_k over the entire interval $(0, \infty)$. For waves of type TH_j it will have the form

$$ik\,\frac{dz_{12}}{dr} - \varepsilon k^2 z_{12} + k^2\left[1 - \frac{1}{\varepsilon}\left(\frac{\nu}{kr}\right)^2\right] = 0. \tag{12}$$

For an ideal Earth it is convenient to choose the point $\bar{r} = a = 6370$ km, i.e., on the Earth’ s surface; then (2) takes the form $z_{12}(a, \infty; \nu) = 0$. Its left-hand side is computed by running through (12) on a computer the radiation condition $z_{12} \approx \sqrt{\varepsilon - (\nu/kr)^2}/\varepsilon$. Since we cannot place the initial point at ∞, we have to assume that this condition applies at a point r_∞ sufficiently far from the lower boundary of the ionosphere, where it is satisfied with some error. The error in ν_j arising in this way was estimated by us by Olver’ s method [1].

Table 1

n	$\alpha_3^{(n)}$	$\beta_3^{(n)}$	$u\left(a, 0; \nu_3^{(n)}\right)$
1	2000	0	$10^{-1}(0,037 + i\,0,69)$
2	2009,52879	30,6664594	$-10^{-1}(0,12 + i\,0,21)$
3	2005,08771	24,2812143	$10^{-3}(0,17 + i\,0,28)$
4	2005,04076	24,3731651	$10^{-5}(0,06 + i\,0,31)$
5	2005,04106	24,3741284	$-10^{-7}(0,44 + i\,0,07)$
6	2005,04105	24,3741273	$-10^{-9}(0,48 + i\,0,01)$

7. The process of running z_{12} from h_∞ to $h = 0$ is represented by the hodograph $u'(h) + iu''(h) = i\varepsilon k z_{12}$, showing how reflections accumulate in the wave leaving into space. Figure 1 gives the hodograph for $f = 60$ kHz

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[1] F. W. Olver, Proc. Cambr. Phil. Soc., **57**, 790 (1961).

and $j = 2$. The main contribution to the reflection is made by the region of large $d\varepsilon/dr$ (55-62 km). Beneath it $u(h)$ moves almost along a circle, which indicates constancy of the reflection coefficient. Only for $h < 20$ km do distortions arise, caused by the adhesion of waves to the concave layers of the ionosphere (the whispering-gallery effect). The number of turns of the hodograph is smaller by one than the number of wavelengths. Table 1, for $j = 3$ and $f = 16$ kHz, shows the rate of convergence of $\nu_3^{(n)}$ to ν_3 by Newton' s method, while Table 2 compares calculations of $\Delta\alpha_j = \alpha_j - ka$ and β_j for $j = 1, 2$ and $f = 10, 16, 25$ kHz, made by the method proposed here and by the semianalytical method of papers [2], [3].

Table 2

f, kHz	method	$\Delta\alpha_j$ ($j = 1$)	β_j ($j = 1$)	$\Delta\alpha_j$ ($j = 2$)	β_j ($j = 2$)
10	p. a.*	$-5,00$	$2,15$	–	–
10	p. v.**	$-5,00632$	$2,14921425$	–	–
16	p. a.	$1,28$	$1,52$	$-39,72$	$8,63$
16	p. v.	$1,28531$	$1,53143243$	$-40,11368$	$8,85235546$
25	p. a.	$8,15$	$1,92$	$-17,1$	$5,91$
25	p. v.	$9,13293$	$1,92289843$	$-17,21134$	$5,92170668$

* p. a. –semianalytical method of papers [4], [5].
** p. v. –purely computational method, accuracy 7 digits.

The calculation of ν_j for tropospheric waveguides [6] is carried out similarly. Integration of (5) on a computer for the case of a SDW for tensor ε confirmed the fact of the stability of run 3.

The analytical part of the work (items 1-4) was performed by P. E. Krasnushkin, and the computational part (items 5-7) by R. B. Baibulatov.

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Note: Figure translations are in progress. See original paper for figures.

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