# ON AN EXACTLY SOLVABLE QUASISPIN THIRRING MODEL

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## Abstract

## Full Text

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**N. N. BOGOLYUBOV, Jr.**

# ON AN EXACTLY SOLVABLE QUASISPIN THIRRING MODEL

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Recently important papers by Thirring ([1,2]) have appeared, devoted to certain aspects of obtaining asymptotically exact results for a quasispin model. In these papers Thirring considers a quasispin system and gives a very interesting direct method for computing the ground state of the system, correlation functions, and Green' s functions. The Hamiltonian of such a system is represented by him in the form

$$H_\Omega = \sum_{p=1}^{\Omega} \varepsilon_p (1 - \sigma_p^z) - \frac{2T_c}{\Omega} \sum_{p=1}^{\Omega} \sigma_p^- \sum_{p'=1}^{\Omega} \sigma_{p'}^+; \tag{1}$$

here $\Omega$ is the number of paired states—a quantity proportional to the volume of the system; $T_c$ is the interaction constant; $\sigma_p^x, \sigma_p^y, \sigma_p^z$ are the Pauli operators. The operators $\sigma_p^+$, $\sigma_p^-$ are expressed in the usual way through $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$.

We shall show that the results obtained by Thirring also follow directly from our general consideration ([3–6]).

In our papers ([3–6]) we considered systems defined by the Hamiltonian

$$H = T - 2\Omega \sum_{\alpha=1}^{s} I_\alpha I_\alpha^+, \tag{2}$$

where $T, I, I^+$ are certain operators satisfying the following additional conditions, the so-called commutation relations:

$$\begin{gathered} |I_\alpha| \le K_1, \qquad |I_\alpha^+ I_\beta - I_\beta I_\alpha^+| \le K_3/\Omega, \\ |T I_\beta - I_\beta T| \le K_2, \qquad |I_\alpha I_\beta - I_\beta I_\alpha| \le K_3/\Omega \\ (1 \le \alpha \le s; \qquad 1 \le \beta \le s), \end{gathered} \tag{3}$$

and it was assumed that the free energy, computed per unit volume, is finite for the Hamiltonian $H = T$,

$$|f_T| \leq K_0, \qquad f_T = -\frac{\theta}{\Omega} \ln \operatorname{Sp} e^{-T/\theta}.$$

Here $K_0, K_1, K_2, K_3$ are certain constants as $\Omega \to \infty$.

Under these conditions it was proved that the difference of the free energies constructed on the basis of the model Hamiltonian (2) and the corresponding approximating Hamiltonian tends to zero as $\Omega \to \infty$, i.e.

$$|f_H - f_{H_B}| \leq \eta(1/\Omega) \to 0 \quad \text{as } \Omega \to \infty.$$

In this inequality $H_B$ is the approximating Hamiltonian

$$H_B = T - 2\Omega \sum_{\alpha=1}^{s} (C_\alpha I_\alpha^+ + C_\alpha^* I_\alpha) + 2\Omega \sum_{\alpha=1}^{s} |C_\alpha|^2. \tag{4}$$

The constants $C_\alpha$ entering it are determined from the condition of an absolute minimum of the function

$$f_{H_B}(C) = -\frac{1}{\Omega} \theta \ln \operatorname{Sp} e^{-H_B/\theta}$$

in the domain of all complex variables $C = (C_1, \dots, C_s)$.

Using the minimizing values $C$, we compute the free energy per unit volume $\Omega$ by the method of the approximating Hamiltonian ([7,8])

$$f_{H_B} = \min_{(c)} f_{H_B}(C).$$

It is clear that the Thirring model belongs precisely to this type of model Hamiltonians. Indeed, taking as the operators $I, I^+, T$ the expressions

$$I = \frac{\sqrt{2T_c}}{\Omega} \sum_{p=1}^{\Omega} \sigma_p^+, \qquad I^+ = \frac{\sqrt{2T_c}}{\Omega} \sum_{p=1}^{\Omega} \sigma_p^-, \tag{5}$$

$$T = \sum_{p=1}^{\Omega} \varepsilon_p (1 - \sigma_p^z), \tag{6}$$

and substituting them in the case $s = 1$ into (2), we arrive at the model system (1).

Let us note that in expression (6), for convenience of calculation, it is usually assumed that $\varepsilon_p$ does not depend on $p$ and is taken to be a constant; systems are also considered for which

$$T = -\varepsilon \sum_{p=1}^{\Omega} \sigma_p^z.$$

For our approach this is immaterial. The commutation relations (3) for the model (1):

$$|I| \le a_1, \qquad |I^+ I - I I^+| \le a_3/\Omega, \qquad |TI - IT| \le a_2, \qquad |f_T| \le a_0,$$

where $a_0, a_1, a_2, a_3$ are certain constants as $\Omega \to \infty$, are obviously fulfilled.

Thus, we have shown that our generalized model problem includes the Thirring model system and therefore our general results ([3–6]) will be applicable to the model (1).

For example, it can be shown that the free energy for the model system (1) is computed exactly as $\Omega \to \infty$. To this end we present the approximating Thirring Hamiltonian for the system (1),

$$H_B = -\varepsilon \sum_{p=1}^{\Omega} \sigma_p^z - 2T_c \sum_{p=1}^{\Omega} \{\sigma_p^+ \langle \sigma_p^- \rangle_B + \langle \sigma_p^+ \rangle_B \sigma_p^-\} + 2T_c \Omega \langle \sigma_p^+ \rangle_B \langle \sigma_p^- \rangle_B.$$

Here $\langle ... \rangle_B$ denotes the usual statistical averaging with respect to the Hamiltonian $H_B$,

$$\langle \rho_p \rangle = \mathrm{Sp}\, e^{-H_B/\theta} \sigma_p^- / \, \mathrm{Sp}\, e^{-H_B/\theta}$$

is a quantity independent of $p$.

Let us also note that the approximating Thirring Hamiltonian is equal to our Hamiltonian (4) for $s = 1$. Its diagonalization for a spin system is most conveniently carried out by means of a rotation transformation, and in this way the free energy $f_{H_B}$ can be calculated; moreover, these arguments for spin $\sigma = 1/2$ can be generalized to the case of a system with arbitrary spin value. In this case, applying the method we have developed, one can establish the equality of the free energies for the model and approximating Hamiltonians as $\Omega \to \infty$.

As was shown in our work ([5,6]), analogous results are also obtained for correlation functions and Green' s functions.

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Academy of Sciences of the USSR

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## CITED LITERATURE

1. W. Thirring, A. Wehrl, Comm. math. Phys., **4**, 303 (1967).

2. W. Thirring, Preprint, On the Mathematical Structure of the BCS Model –II, 1967.

3. N. N. Bogoliubov (Jr.), Ukr. matem. zhurn., No. 3 (1965).

4. N. N. Bogoliubov Jr., Physica, **32**, 933 (1966).

5. N. N. Bogoliubov (Jr.), DAN, **168**, No. 4, 766 (1966).

6. N. N. Bogoliubov (Jr.), The Correlation Functions in the Theory of Superconductivity, preprint of the Institute of Theoretical Physics, 1967, Kiev.

7. N. N. Bogoliubov, D. P. Zubarev, Yu. A. Tserkovnikov, DAN, **117**, 788 (1957).

8. N. N. Bogoliubov, Physica, **26**, Suppl. 1 (1960).

*Note: Figure translations are in progress. See original paper for figures.*