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Abstract

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ON THE STRUCTURE OF PRIMARY IDEALS IN RINGS OF FUNCTIONS INTEGRABLE WITH AN INCREASING WEIGHT ON THE HALF-LINE

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Denote by $\mathcal{L}_\varphi(R)$ the Banach space of functions defined on the real axis R with norm

$$\|f\| = \int_{-\infty}^{\infty} |f(t)|\varphi(t)\,dt,$$

where the weight $\varphi(t)$ satisfies the following conditions:

1. $\varphi(t) \geqslant 1.$
2. $\varphi(t+\tau) \leqslant \varphi(t)\varphi(\tau), \quad -\infty < t, \tau < \infty.$
3. $$\int_{-\infty}^{\infty} \frac{\ln \varphi(t)}{1+t^2}\,dt < \infty.$$

Let, further, $\mathcal{L}_\varphi^\infty(R)$ be the space conjugate to $\mathcal{L}_\varphi(R)$. It consists of functions $g(t)$ for which

$$\|g\| = \operatorname{ess\,sup} |g(t)|/\varphi(t) < \infty.$$

By virtue of conditions 1–3, after introducing the convolution operation

$$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\,d\tau$$

the space $\mathcal{L}_\varphi(R)$ becomes a commutative normed regular ring. The space of maximal ideals of this ring is homeomorphic to the real axis, so that every maximal ideal $M(x_0)$ consists of all functions $f(t) \in \mathcal{L}_\varphi(R)$ whose Fourier transform vanishes at the point x_0 (^{1}).

The question of the structure of the primary ideals of the ring $\mathcal{L}_\varphi(R)$ is considerably more delicate. In view of the absence of a unit in $\mathcal{L}_\varphi(R)$, one must, generally speaking, consider two types of primary ideals: primary ideals $I(\infty)$ corresponding to the infinitely distant point of the real axis, i.e. proper closed ideals which are not contained in any maximal ideal $M(x)$, and primary ideals $I(x_0) \subset M(x_0)$, i.e. proper closed ideals not contained in any maximal ideal $M(x)$, $x \neq x_0$, nor in any primary ideal $I(\infty)$. As Berling showed (^{2}), for the rings $\mathcal{L}_\varphi(R)$ Wiener' s approximation theorem in its formulation for the case $\varphi(t) \equiv 1$ is valid, and consequently (see (^{1}), p. 237), in them there are no primary ideals corresponding to the infinitely distant point.

In studying the primary ideals $I(x)$ (one may assume, without loss of generality, that $x = 0$) it is immediately found that the sets $I_k(0)$ of functions from $\mathcal{L}_\varphi(R)$

$$I_k(0) = \left\{ f : \int_{-\infty}^{\infty} f(t) t^j \, dt = 0, \;\; j = 0, \ldots, k \right\} \tag{1}$$

form a chain of primary ideals belonging to the maximal ideal $M(0) = I_0(0)$, finite or infinite depending on the rate of growth of $\varphi(t)$ as $|t| \to \infty$. Moreover, it turns out that in the case of an infinite chain $\{I_k(0)\}$ the annihilator $I^\perp \subset \mathcal{L}_\varphi(R)$ of any primary ideal $I \subset \bigcap I_k(0)$ consists of functions extendable to the whole complex plane as entire functions of zero degree (see, for example, (^{3})).*

In the present note we consider the subring $\mathcal{L}_\varphi(R^+)$ of the ring $\mathcal{L}_\varphi(R)$, which consists of functions equal to zero on the left half-axis. In the note (^{4}) all primary ideals of the ring $\mathcal{L}_\varphi(R^+)$ were described in the case when $\varphi(t) \equiv 1$. The results given there are readily carried over to the case of the ring $\mathcal{L}_\varphi(R^+)$, where the weight $\varphi(t)$ has no more than power growth as $t \to \infty$. Therefore to conditions 1—3, which the weight $\varphi(t)$ satisfies, we add the condition

$$4. \quad \lim_{t\to\infty} \frac{t^k}{\varphi(t)} = 0 \quad (k = 1, 2, \ldots).$$

The Fourier transform of a function $f(t) \in \mathcal{L}_\varphi(R^+)$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) e^{-itz} \, dt, \qquad \operatorname{Im} z \leqslant 0,$$

is an analytic function in the half-plane $\operatorname{Im} z < 0$ and is infinitely differentiable in the closed half-plane $\operatorname{Im} z \leqslant 0$.

The maximal ideals $M(z_0)$ of the ring $\mathcal{L}_\varphi(R^+)$ may be identified with the points z_0 of the closed lower half-plane; here each maximal ideal $M(z_0)$ consists of all functions $f(t) \in \mathcal{L}_\varphi(R^+)$ whose Fourier transform $F(z)$ vanishes at $z = z_0$.

As in the case $\varphi(t) \equiv 1$ (^{4}), there exist primary ideals of three types: primary ideals $I(\infty)$, corresponding to the infinitely remote point; primary ideals $I(z)$, corresponding to a point z of the open half-plane; and primary ideals $I(x)$, corresponding to a point x of the real axis.

In the article (5) the following approximation theorem was proved in the space $\mathcal{L}_\varphi(R^+)$, where the weight $\varphi(t)$ satisfies conditions 1–3.

Let $\mathfrak{M}$ be a family of functions $\{f_\alpha(t)\}$ of the space $\mathcal{L}_\varphi(R^+)$. For the completeness in $\mathcal{L}_\varphi(R^+)$ of the system of all finite linear combinations of the form

$$\sum_{\alpha,\beta} C_{\alpha\beta} f_\alpha(t-\tau_{\alpha\beta}),$$

where $f_\alpha(t) \in \mathfrak{M}$, $\tau_{\alpha\beta} \geqslant 0$, it is necessary and sufficient that two conditions be fulfilled: 1) there is no interval $(0,\gamma)$, adjoining zero, on which every function in $\mathfrak{M}$ is equal to zero almost everywhere; 2) the Fourier transforms of all functions in $\mathfrak{M}$ do not vanish simultaneously at any point of the closed half-plane $\operatorname{Im} z \leqslant 0$.

This theorem makes it possible to describe completely all primary ideals corresponding to the infinitely remote point and to points of the open half-plane.

For each $\gamma > 0$ introduce the subset $I_\gamma(\infty)$ of functions of the ring $\mathcal{L}_\varphi(R^+)$ that vanish almost everywhere on the interval $(0,\gamma)$. It is clear that $\{I_\gamma(\infty)\}_{\gamma>0}$ is a chain, ordered by inclusion, of primary ideals corresponding to the infinitely remote point. Let $I(\infty)$ be some primary ideal corresponding to the infinitely remote point, and let $(0,\gamma_I)$, $\gamma_I \geqslant 0$, be the maximal interval adjoining zero on which all functions in $I(\infty)$ vanish almost everywhere. The approximation theorem, applied to the family $\{f(t+\gamma_I)\}$ of all functions $f(t) \in I(\infty)$, shows that $I(\infty) = I_\gamma(\infty)$ for $\gamma = \gamma_I$, i.e., that it is true

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* A detailed study of primary ideals in $L_\varphi(R)$ was carried out in (12), though under very stringent additional restrictions on the weight $\varphi(t)$.

Theorem 1. $\{I_\gamma(\infty)\}_{\gamma>0}$ is a maximal chain of primary ideals corresponding to the infinitely remote point. Moreover,

$$\bigcap_\gamma I_\gamma(\infty) = \{0\}.$$

Denote

$$I_n(z_0) = \{f \in M(z_0)\ (\operatorname{Im} z_0 < 0): \ F^{(k)}(z_0) = 0, \quad k = 0, 1, \dots, n\}.$$

$\{I_n(z_0)\}_{n=1}^\infty$ is a chain of primary ideals corresponding to the point z_0, $\operatorname{Im} z_0 < 0$. If one uses the fact that for every function $f(t) \in I_n(z)$, $F(z)/(z-z_0)^n$ is the Fourier transform of some function from $\mathcal{L}_\varphi(R^+)$, then from the approximation theorem one obtains

Theorem 2. $\{I_n(z_0)\}_1^\infty$ is a maximal chain of primary ideals belonging to the maximal ideal $M(z_0)$. Moreover,

$$\bigcap_1^\infty I_n(z_0) = \{0\}.$$

The situation becomes considerably more complicated in the study of primary ideals corresponding to a point x of the real axis. (Since the mapping $f \mapsto fe^{-itx}$ is an isomorphic mapping of the primary ideal $I(x) \subset M(x)$ onto the primary ideal $I(0) \subset M(0)$, in what follows one may restrict oneself to the study of primary ideals contained in the maximal ideal $M(0)$.)

In contrast to the case of the whole axis, where a discrete chain of primary ideals of the form (1) was naturally associated with the point 0, in the space $\mathcal{L}_\varphi(R^+)$ with the point 0 there are associated, generally speaking, two chains of primary ideals. One of them, the discrete one, $\{I_n(0)\}_1^\infty$, consists of the primary ideals

$$I_n(0) = \left\{ f \in M(0) : \int_0^\infty f(t)t^k\,dt = 0, \quad k = 0, 1, \dots, n \right\}.$$

The other, continuous and ordered by inclusion, chain $\{I_\alpha(0)\}_{\alpha>0}$ consists of the primary ideals

$$I_\alpha(0) = \left\{ f \in M(0) : \int_0^\infty f(t)g(t)\,dt = 0, \quad \forall g(t) \in B_{1/2,\alpha,\varphi} \right\}. \tag{2}$$

By $B_{1/2,\alpha,\varphi}$ we denote the subspace of $\mathcal{L}_\varphi^\infty(R^+)$ consisting of functions that coincide almost everywhere with functions extendable to the entire complex plane as entire functions of order $1/2$ and of type less than or equal to α.

Let us consider two primary ideals

$$\tilde{I}(0) = \bigcap_{n>1} I_n(0), \qquad \underset{\sim}{I}(0) = \overline{\bigcup_{\alpha>0} I_\alpha(0)}.$$

In view of condition 4, the space $\mathcal{L}_\varphi^\infty(R^+)$ contains all polynomials, and therefore every primary ideal of the first chain is a proper primary ideal containing all primary ideals of the second. Thus,

$$\tilde{I}(0) \supset \underset{\sim}{I}(0).$$

The nontriviality of the second chain essentially depends on the rate of growth of the weight $\varphi(t)$ as $t \to \infty$.

Theorem 3. Suppose the weight $\varphi(t)$ satisfies the condition

$$\int_1^\infty \frac{\ln \varphi(t)}{t^{3/2}}\,dt < \infty. \tag{3}$$

Then $\{I_n(0)\}_1^\infty$ and $\{I_\alpha(0)\}_{\alpha>0}$ are strictly inclusion-ordered maximal chains of primary ideals contained in the maximal ideal $M(0)$, in the sense that every primary ideal $I(0) \subset M[0]$ either coincides with one of the ideals of the first or second chain, or

$$\underset{\sim}{I}(0) \subset I(0) \subset \tilde{I}(0).$$

Theorem 4. If the weight $\varphi(t)$ satisfies the condition

$$\int_0^\infty \frac{\ln \varphi(t)}{t^{3/2}}\, dt = \infty, \tag{4}$$

then $I_\alpha(0) = \{0\}$ for every $\alpha > 0$.

Formula (2), which determines the primary ideals of the second chain, makes it possible, from the form of the function $f(t)$, to judge its membership in the primary ideal $I_\alpha(0)$. Moreover, the nontriviality of $I_\alpha(0)$ under condition (3) is by no means obvious. It turns out that the following criterion holds for membership of a function $f(t)$ in the primary ideal $I_\alpha(0)$, making it easy to construct nontrivial functions from $I_\alpha(0)$.

Theorem 5. In order that $f(t)$ belong to the primary ideal $I_\alpha(0)$, it is necessary and sufficient that, for the function

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_0^\infty \frac{f(t)}{t - z}\, dt,$$

analytic in the complex z-plane cut along the positive ray of the real axis, the inequality

$$\overline{\lim_{t\to-\infty}} \left(\frac{\ln |\tilde{f}(t)|}{\sqrt{-t}} \right) \leq -\alpha$$

hold.

The question of the structure of the primary ideals $I(0)$ for which the strict inclusions

$$\underline{I}(0) \subset I(0) \subset \tilde{I}(0) \tag{5}$$

hold is connected with the "irregularity" of the weight $\varphi(t)$. The point is that if the weight $\varphi(t)$ is "regular," then ideals satisfying condition (5) do not exist. Namely, the following theorems hold.

Theorem 6*. Let the weight $\varphi(t)$ satisfy condition (3), and let there exist a sequence of positive polynomials $P_n(t)$, uniformly bounded in the norm of the space $\mathcal{L}_\varphi^\infty(R^+)$, such that $P_n(t) \to \varphi(t)$, $0 \leq t < \infty$. Then $\tilde{I}(0) = I(0)$.

Theorem 7. If the logarithmically convex minorant (see ([6])) of the weight $\varphi(t)$ satisfies condition (4), then $\tilde{I}(0) = \{0\}$.

In conclusion we note that the questions considered in the article are closely connected with two classical problems of analysis: the quasianalyticity of classes of functions analytic in the half-plane $\operatorname{Im} z < 0$ and infinitely differentiable in the closed half-plane (see, for example, ([7])), and weighted approximation by polynomials on the half-axis ([8,9]).

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* It is interesting to compare this result with the theorems of the articles ([10,11]), which contain criteria for the completeness of polynomials in the space of entire functions of zero degree.