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Abstract

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MATHEMATICS

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ON THE GROWTH OF SPECTRAL FUNCTIONS OF GENERALIZED SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH A BOUNDARY CONDITION AT A REGULAR END

(Presented by Academician I. G. Petrovskii, 20 XI 1967)

1. Let $M(x)$ be a nondecreasing function on the interval $[0, L)$ $(0 < L \leq +\infty)$, not identically constant. We shall be concerned with boundary-value problems in which the generalized differential expression

$$\frac{d}{dM(x)}\, y^-(x)$$

will occur (as well as more general ones; see item 4)*. For convenience of reading, let us recall the basic definitions. Any function $y(x)$ to which such a differential expression is to be applied will be regarded as given if, in addition to its values at each point of the interval $[0, L)$, the attached number $y^-(0)$ is also specified (which, for convenience, may be called the left derivative at the point $x = 0$). We shall call a function $u(x)$ a solution of the equation

$$\frac{d}{dM(x)}\, u^-(x) = f(x),$$

if the following conditions are satisfied: a) the function $u(x)$ is absolutely continuous on $[0, L)$; b) at each point $x \in (0, L)$ there exists a left derivative $u^-(x)$; c) the function $u^-(x)$ (with account taken of the attached value) is M-absolutely continuous on $[0, L)$; d) the equality

$$\frac{d}{dM(x)}\, u^-(x) = f(x)$$

holds M-almost everywhere on $[0, L)$.

Consider the boundary-value problem

$$-\frac{d}{dM(x)}\, y^-(x) - \lambda y(x) = 0 \quad (0 \leq x < L), \qquad y^-(0) = m, \qquad y(0) = n, \quad (1)$$

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where m and n are real numbers, at least one of which is nonzero. In accordance with the terminology adopted by us in ([7]), the end $x = 0$, at which the boundary conditions in the boundary-value problem (1) are prescribed, is regular.

We shall retain the definition of the spectral function $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$ of the boundary-value problem adopted in ([8]), § 3. M. G. Krein ([9], [10]) showed that, in the case $n \neq 0$, every spectral function $\tau(\lambda)$ of the boundary-value problem (1) is bounded on $(-\infty, 0)$. As for the behavior of the spectral function $\tau(\lambda)$ as $\lambda \to +\infty$, for $\alpha = -1$ the integral

$$\int_1^{+\infty} \lambda^\alpha \, d\tau(\lambda); \tag{2}$$

always converges ($n \neq 0$). For $\alpha = 0$ this integral converges if and only if $M(+0) > M(0)$. Moreover, as was established by M. G. Krein ([10]), any nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < +\infty)$, constant on $(-\infty, 0)$, and growing as $\lambda \to +\infty$ in such a way that for $\alpha = -1$ the integral (2) converges, is the spectral function of a boundary-value problem of the form (1) with $m = 0,\ n = 1$.

* The theory of such differential expressions is set out in detail in our paper ([8]). References to the literature in which such differential expressions were considered are also given there.

In a number of our papers ([3–7]) the behavior of the spectral function $\tau(\lambda)$ as $\lambda \to +\infty$ was studied in dependence on the behavior of the function $M(x)$ in a neighborhood of the point at which the boundary conditions are prescribed. For example, in ([3,5]) the convergence of the integral (2), for real α, was studied in the case where the boundary condition is prescribed at a regular end. In view of what was said above, the question of its convergence when $n \neq 0$ is of interest only when $-1 < \alpha < 0$. In Sec. 3 of the present paper, in the case where $n \neq 0$, a necessary and sufficient condition for convergence of the integral

$$\int_1^{\infty} \eta(\lambda) \, d\tau(\lambda) \tag{3}$$

for a certain class V of functions $\eta(\lambda)$, including the functions λ^α with $\alpha \in (-1, 0)$, will be formulated in terms of the behavior of the function $M(x)$ as $x \downarrow 0$ (information about this class of functions is given in the next paragraph).

2. Definition 1. We shall assign a nonincreasing positive function $\eta(\lambda)$, $(1 \leq \lambda < \infty)$, to the **class** V if the functions $F(t)$ and $G(t)$, defined for $t \geq 1$ by the equalities

$$F(t) = \inf_{\lambda \geq 1} [\eta(\lambda t)/\eta(\lambda)], \qquad G(t) = \sup_{\lambda \geq 1} [\eta(\lambda t)/\eta(\lambda)]$$

are such that $(t^2 F(t))^{-1} \in \mathcal{L}_1(1, +\infty)$ and $t^{-1} G(t) \in \mathcal{L}_1(1, +\infty)$.

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It is easy to see that the power functions λ^α, $(1 \leq \lambda < \infty)$, with $-1 < \alpha < 0$, belong to the class V, and that no other power functions considered on the interval $[1, +\infty)$ belong to this class. At the same time, the class V is considerably broader than the class of such power functions. In the theory of entire functions, functions of the form $\lambda^{\rho(\lambda)}$ are considered as functions close to powers for large λ, where $\rho(\lambda)$ is the so-called generalized order, i.e., such a function of the real variable λ that $\lim_{\lambda\to+\infty} \rho(\lambda) = \rho(+\infty)$ is a finite number, and $\lambda\rho'(\lambda)\ln\lambda \to 0$ as $\lambda \to +\infty$ (see ([11]), p. 47). It can be shown that any function of the form $\lambda^{\rho(\lambda)}$, where $\rho(\lambda)$ is a generalized order with $\rho(+\infty) \in (-1, 0)$, coincides for large λ with a function belonging to the class V. Moreover, the function $\lambda^{\rho(\lambda)}$, under the indicated conditions, belongs to the class V if it decreases on $[1, +\infty)$; the latter can always be achieved if the function $\rho(\lambda)$ is changed and extended (if it is not defined for all $\lambda \geq 1$) only on a finite interval (see ([11]), p. 49), while thus leaving it a generalized order.

The class V, however, is not exhausted by functions of this form. Thus, for any α and β such that $-1 < \alpha < \beta < 0$, one can construct a function $\eta(\lambda)$ belonging to the class V for which, for all $t > 1$, the lower limit of the ratio $\eta(\lambda t)/\eta(\lambda)$ as $\lambda \to +\infty$ is equal to t^α, and the upper limit to t^β; at the same time, for any function $\eta(\lambda) = \lambda^{\rho(\lambda)}$, where $\rho(\lambda)$ is a generalized order, $\eta(\lambda t)/\eta(\lambda) \to t^{\rho(+\infty)}$ as $\lambda \to +\infty$ (see ([11]), p. 48).

3. Theorem 1. *Let $\tau(\lambda)$ be any spectral function of the boundary-value problem (1), in which $n \neq 0$, and let the function $M(x)$ be such that $M(x) > M(0)$ for every $x \in (0, L)$. Then the interval (3), where $\eta(\lambda) \in V$, converges if and only if, at least for one $l \in (0, L)$, the integral*

$$\int_0^l \frac{1}{N(x)}\,\eta\left(\frac{1}{N(x)}\right) dx, \qquad \text{where} \quad N(x) = \int_0^x [M(s) - M(0)]\,ds$$

converges (and, consequently, for every $l \in (0, L)$ such that $N(l) \leq 1$).

In the case where $n = 0$, the picture changes sharply. Now convergence of the integral (2) can be guaranteed only for $\alpha = -2$, while for $\alpha = -1$ this integral converges if and only if $M(x) = M(0)$

for some $x > 0$. In this connection, if we exclude the latter case, the question of convergence of the integral (2) is of interest only for $-2 < \alpha < -1$, and therefore the question of convergence of the integral (3) will be of interest when $\lambda\eta(\lambda) \in V$.

Theorem 2. If $\tau(\lambda)$ is some spectral function of the boundary-value problem (1), in which $n = 0$, and the function $M(x)$ is such that $M(+0) = M(0)$, but $M(x) > M(0)$ for every $x \in (0, L)$, then the integral (3), where $\lambda\eta(\lambda) \in V$, converges if and only if, for at least one $l \in (0, L)$, the integral

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$$\int_0^{l-0} \frac{1}{(N_0(x))^2} \eta\left(\frac{1}{N_0(x)}\right) dM(x),$$

where

$$N_0(x) = \frac{1}{2}\left(\int_0^{x-0} s\, dM(s) + \int_0^{x+0} s\, dM(s)\right).$$

4. Theorems 1 and 2 remain valid if, in their formulations, the boundary-value problem (1) is replaced by the problem

$$-\frac{d}{dM(x)}\left[y^-(x) - \int_{-0}^{x-0} y(s)\, dQ(s)\right] - \lambda y(x) = 0 \quad (0 \le x < L); \tag{4}$$

$$y^-(0) = m, \qquad y(0) = n,$$

where $Q(x)$ is a real function having bounded variation on each interval $[0, l]$ with $l \in [0, L]$, and $M(x), m$ and n are the same as in problem (1). The definition of a solution of the generalized differential equation from problem (4) is given in (8) (pp. 215, 216 and 181). We note here only that the function $u(x)$ can be a solution of this equation only in the case when, on each open interval of constancy of the function $M(x)$, it satisfies the equation

$$y^-(x) - \int_{\gamma-0}^{x-0} y(s)\, dQ(s) = \text{const}, \tag{5}$$

where γ is some fixed point of this interval.

5. The presence of intervals of constancy and jump points of the function $M(x)$ may lead to the absence of spectral functions for the boundary-value problem (4), so that Theorems 1 and 2, after replacing problem (1) in them by (4), will describe properties of the empty set of functions. In this section we give conditions for the existence of spectral functions of the boundary-value problem (4).

Definition 2. An interval $[\alpha, \beta] \subset [0, L)$ is called MQ-strongly distorting if the following conditions are satisfied: a) in the interval (α, β) there are no points of increase of the function $M(x)$; b) the points $x = \alpha$ and $x = \beta$ are jump points of the function $M(x)$; c) there exists a function $u(x)$, different from identically zero on $[\alpha, \beta]$, absolutely continuous on $[\alpha, \beta]$, vanishing at the endpoints $x = \alpha$ and $x = \beta$, and satisfying equation (5) on the interval (α, β).

Remark. The last condition cannot be satisfied when $Q(x)$ does not decrease on (α, β) (see (8), § 7, item 4).

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Definition 3. We shall say that the boundary-value problem (4) (in particular, (1)) has a left distorting endpoint if the following conditions are satisfied: a) the exact lower bound a_0 of the set of points of increase of the function $M(x)$ is a jump point of this function; b) for at least one (and, consequently, for every) value of λ, the (unique) solution of the differential system (4) vanishes at the point $x = a_0$.

Remark. The second of the conditions means that either $a_0 = 0$ and $n = 0$, or $a_0 > 0$ and there exists an absolutely continuous function $u(x)$ on $[0, a_0]$, satisfying on the interval $(0, a_0]$ equation (5) with $\gamma = 0$ and const $= m$, and at its endpoints the conditions $u(0) = n$, $u(a_0) = 0$; the existence of such a function is impossible if $mn \geq 0$, and $Q(x)$ does not decrease on $[0, a_0)$.

Theorem 3. *In order that, under the conditions imposed on the functions $M(x)$ and $Q(x)$ at the beginning of § 4, the boundary-value problem (4) have at least one spectral function, it is necessary and sufficient that this boundary-value problem have no left-distorting end and that there be not a single MQ-strongly distorting interval $[\alpha, \beta] \subset [0, L)$.*

By virtue of Theorem 3, the boundary-value problem (1) has no spectral function if and only if $-n = a_0 m$ and the M-measure of the point a_0 is positive (in this case the problem has a left-distorting end; by the remark to Definition 2 there can be no strongly distorting intervals here), but the conditions of Theorems 1 and 2 exclude such a possibility.

In the case when $M(x)$ and $Q(x)$ are absolutely continuous functions and $M'(x) = r(x)$, $Q'(x) = q(x)$ almost everywhere on $[0, L)$, the boundary-value problem (4) is equivalent to the boundary-value problem

$$-y'' + q(x)y - \lambda r(x)y = 0 \quad (0 \leq x < L), \qquad y'(0) = m, \qquad y(0) = n. \quad (6)$$

In this case Theorems 1 and 2, after replacing in their formulations the boundary-value problem (1) by (4) (essentially by (6)), if one sets in them $\eta(\lambda) = \lambda^\alpha$, give as a special case the most substantial part of the main theorem from ([4]). We note that, by virtue of Theorem 3, the boundary-value problem (6) has spectral functions even when $r(x)$ vanishes on some intervals. In this connection, the requirement introduced in ([4]) that the function $r(x)$ be positive for all $x \in [0, L)$ can be replaced by the weaker one: $r(x) \geq 0$ for all $x \in [0, L)$, and there is no interval $[0, l)$ $(l > 0)$ on which $r(x) = 0$ almost everywhere.

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* In our paper ([2]), in the formulations of theorems giving sufficient conditions for the existence of spectral functions of boundary-value problems with generalized differential equations such as in problem (4), through oversight the conditions guaranteeing the existence of spectral functions in the presence of intervals of constancy and jump points of the functions $M(x)$ were omitted. In paper ([8]) this error, in the case of a singularity of the end at which the boundary conditions are prescribed, was corrected, but the sufficient conditions given there (see ([8]), Theorems 11, 12, 13) were too restrictive. Still more restrictive conditions are imposed in the book of F. V. Atkinson ([1]).

Note: Figure translations are in progress. See original paper for figures.

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