DIFFERENCE METHODS FOR SOLVING AN ILL-POSED CAUCHY PROBLEM MODELING THE FLOW OF AN IDEAL GAS IN A NOZZLE

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Abstract

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DIFFERENCE METHODS FOR SOLVING AN ILL-POSED CAUCHY PROBLEM MODELING THE FLOW OF AN IDEAL GAS IN A NOZZLE

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1. As is known, in the problem of the irrotational flow of an ideal gas in a Laval nozzle two problems arise: the direct and the inverse. Each of them, while possessing certain advantages in its formulation, at the same time contains considerable difficulties in its solution. The direct problem, which is more natural, leads to the consideration of a boundary-value problem for the system of equations of an ideal gas; the proof of the well-posedness (in the usual sense) of this problem poses considerable difficulties because of the nonlinearity of the system of equations under consideration (in particular, even the uniqueness of the solution of the direct problem has not been proved rigorously). The inverse problem in the subsonic and supersonic parts of the nozzle leads to the consideration of the Cauchy problem for a system of first-order differential equations having complex characteristics. Although it has a unique solution in the class of analytic functions, it is not well-posed (in the sense of Hadamard), since without special additional assumptions it is impossible to ensure continuous dependence of the solution on the initial data. When attempts are made to solve the inverse problem by one or another approximate method of discrete character (using difference schemes, the Galerkin method, etc.), the ill-posedness of the problem manifests itself immediately in the form of rapid accumulation of computational error. However, the application of one or another regularization method (see (1)) makes it possible in some cases to overcome this difficulty. In the work of E. G. Shifrin, the regularization proposed in (2) was used; in work (4) the regularizing effect was obtained by a special choice of the difference mesh.

The regularization method studied theoretically in (3), by adding fictitious viscosities, has been successfully applied to the computation of particular flows of an ideal gas (see (5)). However, the specificity of the inverse problem and the appearance, in experimental computations, of boundary effects lead to the necessity of considering a somewhat different linear model, one that better reflects the essence of the basic gas-dynamical problem.

In work (3) the natural domain in which the problem was considered was a strip.

In problems concerning the flow of an ideal gas in a nozzle, where it is necessary to consider subsonic and supersonic flows simultaneously, the natural domain in which the ill-posed Cauchy problem should be studied is an angle. We proceed to the consideration of the model problem.

2. A good model of the inverse problem of the theory of the Laval nozzle in the subsonic and supersonic regions is the Cauchy problem for the Cauchy–Riemann equation in an angular domain. Let $S_\alpha = \{(r,\varphi) :\ 0 < r < \infty,\ 0 < \alpha < \varphi\}$ be a sector of aperture α in the (x,t)-plane ((r,φ) are the polar coordinates corresponding to the point (x,t)). We consider the Cauchy problem for the Cauchy–Riemann equation in polar coordinates

$$\partial u/\partial\varphi = -ir\,\partial u/\partial r, \qquad (r,\varphi) \in S_\alpha, \tag{1}$$

$$u(r,0) = g(r), \qquad r > 0. \tag{2}$$

In the usual sense, problem (1), (2) is ill-posed, like every Cauchy problem for elliptic equations. However, under the additional a priori assumption

$$\|u(\varphi)\|_s^2 = \int_0^\infty \left(|u(r,\varphi)|^2 + \left| \left(r\frac{\partial}{\partial r}\right)^s u(r,\varphi)\right|^2 \right) r^{-1}\,dr \leqslant M_s^2 < \infty, \tag{3}$$

$$0 \leqslant \varphi \leqslant a, \qquad s \geqslant 0,$$

problem (1), (2) becomes well-posed in the sense of A. N. Tikhonov (^{1}). Its conditional well-posedness follows from the following theorem.

Theorem 1. For every solution $u(r,\varphi)$ of equation (1), the inequality

$$\|u(\varphi)\|_s \leqslant \|u(0)\|_s^{1-\varphi/a} \|u(a)\|_s^{\varphi/a} \tag{4}$$

holds.

Inequality (4) means logarithmic convexity, with respect to φ, of the function $\|u(\varphi)\|_s$. From an algorithmic point of view, the most convenient apparatus for the numerical solution of problem (1), (2) is provided by difference methods. We proceed to the consideration of the difference problem.

3. Let $S_{\alpha,h} = \{(r,\varphi) : \varphi = ph,\ r = e^{qh},\ 0 \leqslant ph \leqslant a,\ -\infty < qh < \infty,\ p$ and q integers$\}$ be a difference grid in the sector S_α. Consider the difference Cauchy problem

$$U_h(r,\varphi+h) = \sum_{\rho>0} \alpha_h(\rho) U_h(r\rho^{-1},\varphi)h = C(h) * U_h(r,\varphi), \tag{5}$$

$$U_h(r,0) = g(r), \qquad r > 0, \tag{6}$$

where r, ρ, and φ vary, respectively, on the radial and angular parts of the grid $S_{\alpha,h}$ in the sector S_α.

Strengthening the a priori assumption (3), we shall assume in this section that the solution of problem (1), (2) exists for $\gamma a \leqslant \varphi \leqslant a$, $\gamma < 0$, and

$$\|u(\varphi)\|_s \leqslant M_s, \qquad \gamma a \leqslant \varphi \leqslant a. \tag{3'}$$

Denote by $S(h,\omega)$ the difference Mellin transform of the function $\alpha_h(\rho)$, defined by the formula

$$S(h,\omega) = \sum_{\rho>0} \alpha_h(\rho)\rho^{-i\omega}h \qquad (\omega = \bar{\omega}), \tag{7}$$

where the summation is over the radial part of the grid $S_{\alpha,h}$.

Definition 1. The difference scheme (5) is called **γ-stable** if the function $S(h,\omega)$ satisfies the inequalities

$$|S^{\varphi/h}(h,\omega)| \leqslant \begin{cases} e^{\varphi\omega+c\varphi}, & \omega \geqslant 0, \\ e^{\gamma\varphi\omega+c\varphi}, & \omega < 0, \end{cases} \tag{8}$$

where $\varphi = mh$, $0 \leqslant \varphi \leqslant a$, and c is a constant.

Definition 2. The difference scheme (5) is called **q-approximating equation** (1) if, for any finite on $(0,\infty)$, infinitely differentiable function $g(r)$, the inequality

$$\left\| \left(C(h) - \sum_{k=0}^{q} \frac{h^k}{k!} \left(-ir\frac{\partial}{\partial r}\right)^k \right) g \right\|_0 \leqslant C_q h^{q+1} \|g\|_{q+1}. \tag{9}$$

holds.

For solutions of problem (5), (6), define a difference analogue of the norm $\|\cdot\|_s$ as follows:

$$\|U_h\|^2_{s,h} = h \sum_{r>0} \left(|U_h|^2 + \left|(rD_r)^{sU}_h\right|^2 \right), \tag{10}$$

where $D_r u = (u(r+\Delta r) - u(r))/\Delta r$, $r = e^{ph}$, $\Delta r = r(e^h - 1)$, p is an integer.

In what follows we shall also need the operator $\overline{D}_r$, defined by the equality $\overline{D}_r u = (u(r) - u(r-\Delta r))/\Delta r$.

Theorem 2. Suppose that the difference scheme (5) q-approximates equation (1) and is γ-stable. Then the solution $U_h(r, \varphi)$ of problem (5), (6) converges in the norm $\| \cdot \|_{s-q-1,h}$ to the solution $u(r, \varphi)$ of problem (1), (2), satisfying the a priori assumption (3′) with the same number γ as in (8). The rate of convergence is determined by the inequality

$$\|u(\varphi) - U_h(\varphi)\|_{s-q-1,h} \leq C_q \varphi e^{c\varphi} h^{qM}_s, \tag{11}$$

where C_q is a constant depending only on q, c is a constant, $0 \leq \varphi \leq \alpha$, $\varphi = mh$, and M_s is the same quantity as in (3′).

Thus, as follows from Theorem 2, a γ-stable difference scheme approximating equation (1) is a regularizer in the sense of A. N. Tikhonov (see (1′)) for problem (1), (2) under the a priori assumption (3′). The regularization parameter is the mesh step h of the grid $S_{\alpha,h}$ in the angular direction.

4. In this section we indicate other regularizers for problem (1), (2) under the weaker and more natural a priori assumption (3). Let $K_h(r)$ be a function of the continuous parameter h and the discrete argument r, varying on the radial part of the grid $S_{\alpha,h}$; suppose that the difference Mellin transform of the function $K_h(r)$, defined by formula (7), is the function $\widehat{K}_h(\omega)$ (periodic with period $2\pi/h$) equal to

$$\widehat{K}_h(\omega) = \begin{cases} 1, & |\omega| < q|\ln h|/2\alpha, \\ 0, & q|\ln h|/2\alpha < |\omega| < 2\pi/h, \end{cases} \tag{12}$$

where q is the order of approximation of the difference scheme (5). Put

$$\psi_h(r) = h \sum_{\rho > 0} K_h(r\rho^{-1}) \varphi(\rho). \tag{13}$$

Theorem 3. The solution U_h of the Cauchy problem (5), (13), where the difference scheme (5) is assumed to q-approximate equation (1) and to be γ-stable with arbitrary γ, converges as $h \to 0$ to the solution $u(r, \varphi)$ of (1), (2), satisfying the a priori assumption (3). The rate of convergence is determined by the inequality

$$\|u(\varphi) - U_h(\varphi)\|_{s-q-1,h} \leq C(1 + |\ln h|)^{-(q+1)} \left[h^{q\varphi/2\alpha} \|u(0)\|_s + h^{q(\alpha-\varphi)/2\alpha} \|u(\alpha)\|_s \right], \tag{14}$$

where C is a constant depending only on q and α.

The inconvenience of the regularizer defined by formulas (5), (13) consists in the fact that the averaging operation (13) has an essentially nonlocal character.

Therefore we now indicate another difference regularization process, free from this drawback.

Consider the implicit difference scheme

$$[1-\varepsilon(h) r D_r r \overline{D}_r] U_h(r, \varphi+h) =$$

$$= [1-\varepsilon(h) r D_r r \overline{D}_r] U_h(r, \varphi) + C_1(h) U_h(r, \varphi), \tag{15}$$

where $\varepsilon(h) = a\alpha/|\ln h|$ (α is the size of the angle in which the solution of problem (1), (2) exists, a is a certain positive constant), the operators D_r and $\overline{D}_r$ are defined above, and the difference operator $C_1(h)$ is an operator of convolution type in r,

$$C_1(h) U_h(r, \varphi) = \sum_{\rho>0} \alpha(\rho) U_h(r\rho^{-1}, \varphi), \tag{16}$$

satisfying the requirement: $1 + C_1(h)$ is a γ-stable difference operator with arbitrary γ, q-approximating equation (1) with arbitrary $q > 0$.

Theorem 4. There exists a positive number a_0, depending only on the coefficients $a(\rho)$ of scheme (16), such that for $a \geq a_0$ the solution $U_h(r, \varphi)$ of the system of equations (15) with initial condition (6) converges, as $h \to 0$, in the norm $\|\cdot\|_{s-2,h}$ to the solution $u(r, \varphi)$ of problem (1), (2), satisfying the a priori assumption (3).

The rate of convergence is determined by the inequality:

$$\|u(\varphi) - U_h(\varphi)\|_{s-2,h} \leq c\varphi\varepsilon(h) M_s, \qquad 0 \leq \varphi \leq a, \qquad \rho = mh. \tag{17}$$

If the initial function $g(r)$ is infinitely differentiable and $\|g^i\|_k \leq c^k \Gamma(k/\beta)$ ($0 < \beta \leq 1,\ c$ is a constant, Γ is the gamma function), then in the difference scheme (15), and correspondingly in estimate (17), one may take $\varepsilon(h) = \mathrm{const} \cdot h^\beta$.

In conclusion, we note that all the results of the present note carry over without difficulty to systems of differential equations of the form

$$\partial u/\partial\varphi = Ar\,\partial u/\partial r, \qquad 0 < r < \infty, \qquad 0 < \varphi < a,$$

where A is a constant matrix.

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[1] A. N. Tikhonov, DAN, 151, No. 3 (1963).
[2] L. S. Frank, L. A. Chudov, Collection: *Numerical Methods in Gas Dynamics*, vol. IV, Moscow, 1964.
[3] L. S. Frank, UMN, 19, issue 5 (119) (1964).
[4] U. G. Pirumov, *Izv. AN SSSR, Mechanics of Fluid and Gas*, No. 5 (1967).
[5] L. S. Frank, V. N. Suvorova, Communication at the III All-Union Congress on Theoretical and Applied Mechanics, Moscow, January 1968.

Note: Figure translations are in progress. See original paper for figures.

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