A HYDRODYNAMIC NONSTATIONARY MODEL OF AN INHOMOGENEOUS OCEAN

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Abstract

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A HYDRODYNAMIC NONSTATIONARY MODEL OF AN INHOMOGENEOUS OCEAN

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In paper (1) a stationary model of an inhomogeneous ocean is proposed. Let us now pass to a nonstationary model. The initial system of equations, the boundary and initial conditions are as follows:

$$-\frac{\partial u}{\partial t} + A_z \frac{\partial^2 u}{\partial z^2} + \Omega v = -g\frac{\partial \zeta'}{\partial x} + \frac{g}{\rho}\int_0^z \frac{\partial \rho}{\partial x}\,dz,$$
$$-\frac{\partial v}{\partial t} + A_z \frac{\partial^2 v}{\partial z^2} - \Omega u = -g\frac{\partial \zeta'}{\partial y} + \frac{g}{\rho}\int_0^z \frac{\partial \rho}{\partial y}\,dz; \tag{1}$$

$$w = \int_z^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz; \tag{2}$$

$$\partial S_x/\partial x + \partial S_y/\partial y = \partial \zeta/\partial t; \tag{3}$$

$$\zeta' = \zeta - p_a/g\rho; \tag{4}$$

$$\iint \zeta\, dx\, dy = 0; \tag{5}$$

$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} + v\frac{\partial \tau}{\partial y} + w\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z}\left(\varkappa_{\tau z}\frac{\partial \tau}{\partial z}\right) + \varkappa_{\tau l}\Delta\tau,$$
$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\varkappa_{sz}\frac{\partial s}{\partial z}\right) + \varkappa_{sl}\Delta s; \tag{6}$$

$$\rho = f_\rho(\tau, s, z); \tag{7}$$

$$\tau = f_\tau(a_{1\tau}, a_{2\tau}, a_{3\tau}, z), \qquad s = f_s(a_{1s}, a_{2s}, a_{3s}, z); \tag{8}$$

$$\text{for } z = \zeta \qquad A_z \partial u/\partial z = -T_x/\rho, \qquad A_z \partial v/\partial z = -T_y/\rho; \tag{9}$$

$$p = p_a; \tag{10}$$

$$\varkappa_{\tau z} \partial \tau/\partial z = \Gamma_\tau^0, \qquad \varkappa_{sz} \partial s/\partial z = \Gamma_s^0; \tag{11}$$

$$\text{for } z = H \qquad u = v = 0; \tag{12}$$

$$\partial \tau/\partial z = 0, \qquad \partial s/\partial z = 0; \tag{13}$$

$$\text{on the contour } L \qquad S_n = 0; \tag{14}$$

$$\partial \tau/\partial n = 0, \qquad \partial s/\partial n = 0; \tag{15}$$

$$\text{for } t = 0 \qquad u = v = 0; \tag{16}$$

$$\zeta = 0; \tag{17}$$

$$\tau = \text{const}, \qquad s = \text{const}; \tag{18}$$

$$T_x = T_y = 0; \tag{19}$$

$$p_a = \text{const}; \tag{20}$$

$$\Gamma_\tau^0 = 0, \qquad \Gamma_s^0 = 0. \tag{21}$$

In equations and conditions (1)–(21), u, v, w are the components of the current velocity along the Cartesian coordinate axes X, Y, Z, directed correspondingly –

respectively to the east, north, and vertically downward; ζ is the level; ζ' is the dynamic level; g is the acceleration of gravity; ρ is the density of seawater, τ is temperature; s is salinity; t is time; A_z is the coefficient of vertical exchange of momentum; $\varkappa_{\tau z}$, $\varkappa_{sz}$ and $\varkappa_{\tau l}$, $\varkappa_{sl}$ are coefficients of turbulent diffusion; T_x, T_y

are the components of the tangential wind stress; p_a is atmospheric pressure; $\Gamma_\tau^0, \Gamma_s^0$ are quantities characterizing the fluxes of heat and salt through the ocean surface; H is depth; σ is the surface; L is the contour of the closed basin and n is the direction of the normal to it;
$S_x = \int_0^H u\,dz,\ S_y = \int_0^H v\,dz$ are the components of the total flux; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. The known quantities are the functions of the horizontal coordinates and time $T_x, T_y, p_a, \Gamma_\tau^0, \Gamma_s^0, \varkappa_{\tau z}^0, \varkappa_{sz}^0, A_z$ and the constants $\varkappa_{\tau l}, \varkappa_{sl}$. The functions f are also assumed known. The quantities a_i, depending on the horizontal coordinates and time, are sought in solving the problem.

At the initial instant of time, the temperature and salinity in the ocean are constant, motion is absent, the density depends only on the vertical coordinate (the result of compressibility), and there is no wind and no flux of heat or salt. Wind, a horizontal gradient of atmospheric pressure, and fluxes of heat and salt through the ocean surface arise instantaneously. The ocean is set in motion, and a process of mutual adjustment of the fields of level, current velocity, temperature, salinity, and seawater density takes place.

A characteristic feature of the stated formulation of the problem is the involvement of models of temperature and salinity of seawater (see (1-4)). Owing to this, first, the need to prescribe the laws of variation of the coefficients $\varkappa_{\tau z}, \varkappa_{sz}$ along the vertical is eliminated, and, second, the problem is reduced to solving a system of equations for several functions that do not contain the coordinate z. We note that, as a rule, the solution depends on the variation of the coefficients $\varkappa_{\tau z}, \varkappa_{sz}$ along the vertical to a considerably greater degree than on the variation of the coefficient A_z in the same direction.*

Integrate equation (6) with respect to z from $z = \zeta(x, y, t)$ to $z = H(x, y)$. Taking into account that at the ocean bottom $w = 0$, while at its surface

$$w = u\,\frac{\partial \zeta}{\partial x} + v\,\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial t},$$

and taking account of conditions (11), (13), we obtain

$$\frac{\partial}{\partial t}\int_\zeta^H \tau\,dz + \frac{\partial}{\partial x}\int_\zeta^H \tau u\,dz + \frac{\partial}{\partial y}\int_\zeta^H \tau v\,dz = -\Gamma_\tau^0 + \varkappa_{\tau l}\int_\zeta^H \Delta\tau\,dz. \qquad (22)$$

The physical meaning of equation (22) and of the analogous equation for salinity is clear: the change in the amount of heat (salt) in a unit column extending from the ocean surface to the bottom occurs due to horizontal advection, flux through the ocean surface, and horizontal turbulent diffusion.

Taking the lower limit of integration in equation (22) and in the analogous equation for salinity to be zero, we obtain

$$\frac{\partial}{\partial t}\int_0^H \tau\,dz + \frac{\partial}{\partial x}\int_0^H \tau u\,dz + \frac{\partial}{\partial y}\int_0^H \tau v\,dz = -\Gamma_\tau^0 + \varkappa_{\tau l}\int_0^H \Delta\tau\,dz,$$

$$\frac{\partial}{\partial t}\int_0^H s\,dz + \frac{\partial}{\partial x}\int_0^H su\,dz + \frac{\partial}{\partial y}\int_0^H sv\,dz = -\Gamma_s^0 + \varkappa_{sl}\int_0^H \Delta s\,dz. \qquad (23)$$

* Outside the equatorial zone of the ocean, the direct influence of vertical exchange of momentum is practically manifested only in the thin Ekman layers at the ocean surface and bottom, whereas the direct influence of vertical diffusion of heat extends at least to the depth of the main thermocline.

To solve the problem we have a system of equations, boundary and initial conditions (1)–(5), (7)–(10), (12), (14), (16)–(20). Conditions (15) are "softened" :

$$\text{on the contour } L \qquad \frac{\partial}{\partial n}\int_0^H \tau\,dz = 0, \qquad \frac{\partial}{\partial n}\int_0^H s\,dz = 0. \qquad (24)$$

Let us now restrict ourselves to the case where the fields under study—current velocity, temperature, salinity, density, and level—vary slowly in time*. In other words, let us consider the problem of a large-scale nonstationary ocean circulation, averaged in time and representing a certain background on which much more rapidly varying processes are already developing, associated with the real variability of external factors such as wind, static nonuniformity of atmospheric pressure, etc.

Since the physical fields under consideration vary slowly in time, the term $\partial\zeta/\partial t$ in equation (3) may be neglected ([5])**, and an integral stream function ψ may be introduced by defining it as follows:

$$S_x = -\partial\psi/\partial y, \qquad S_y = \partial\psi/\partial x, \qquad (25)$$

$$\text{on the contour } L \qquad \psi = 0. \qquad (26)$$

Let us now consider two cases:

1. The current varies so slowly that the first terms on the left-hand side of equations (1) may be neglected***. Then from these equations, conditions (9), (12), and formulas (25) we obtain

$$u = N_1 T_x + M_1 T_y + \Lambda_1 \partial\psi/\partial x - \Theta_1 \partial\psi/\partial y - \Theta_1 S_x^* - \Lambda_1 S_y^* + u^*,$$

$$v = -M_1 T_x + N_1 T_y + \Theta_1 \partial\psi/\partial x + \Lambda_1 \partial\psi/\partial y + \Lambda_1 S_x^* - \Theta_1 S_y^* + v^*; \qquad (27)$$

$$\partial\zeta/\partial x = -m' T_x + n' T_y + \nu'(-\partial\psi/\partial y - S_x^*) - \lambda'(\partial\psi/\partial x - S_y^*),$$

$$\partial\zeta/\partial y = -n' T_x - m' T_y + \lambda'(-\partial\psi/\partial y - S_x^*) + \vartheta'(\partial\psi/\partial x - S_y^*); \qquad (28)$$

$$\vartheta' \Delta\psi + (\partial\vartheta'/\partial x + \partial\lambda'/\partial y)\,\partial\psi/\partial x + (\partial\vartheta'/\partial y - \partial\lambda'/\partial x)\,\partial\psi/\partial y =$$

$$= \mathrm{div}(n'T + \lambda' S^*) + \mathrm{rot}_z(m'T + \vartheta' S^*). \qquad (29)$$

The values of the coefficients and functions marked with an asterisk are given in work ([1]). If formulas (7), (8) are substituted into the expressions for S_x^*, S_y^*, and then the result is substituted into equation (29), we obtain an equation containing no z and relating the functions ψ, a_i. Two other equations for these functions are obtained by still taking formulas (7), (8) into account and substituting expressions (27) into equations (23). Solving, by one method or another, the system of the three indicated equations and using conditions (26), (15), we find the integral stream function ψ and 6 functions a_i. Thus, all the main calculations are carried out within the framework of a plane problem. After this problem has been solved, the functions u, v, w, τ, s, ρ at any time and at any point in space are easily calculated from formulas (27), (2), (8), (7). The slopes of the level are calculated from formulas (28), after which, using conditions (5), the level ζ itself is also calculated.

2. The current does not vary so slowly that the first terms on the left-hand side of equations (1) could be neglected. In this case we proceed as follows. We introduce a time step Δt and write equations (1) in finite-difference form for the velocity components–

* For example, from decade to decade, from season to season, and from year to year.
** In addition, the dynamic level ζ' may be identified with the level ζ.
*** We note that the estimate of the terms should be carried out not in equations (1), but in the vorticity equation, which can be obtained from equations (1).

find u, v at the $n+1$-st instant of time ($r = 1/\Delta t$):

$$
\begin{aligned}
-ru^{n+1} + A_z \frac{\partial^2 u^{n+1}}{\partial z^2} + \Omega v^{n+1} &= -g \frac{\partial \zeta'^{n+1}}{\partial x} + \frac{g}{\rho} \int_0^z \frac{\partial \rho^{n+1}}{\partial x}\, dz - ru^n, \\
-rv^{n+1} + A_z \frac{\partial^2 v^{n+1}}{\partial z^2} - \Omega u^{n+1} &= -g \frac{\partial \zeta'^{n+1}}{\partial y} + \frac{g}{\rho} \int_0^z \frac{\partial \rho^{n+1}}{\partial y}\, dz - rv^n.
\end{aligned} \tag{30}
$$

Equations (30) are formally analogous to the equations of steady motion with allowance for lateral friction proportional to the horizontal current velocity ([6]). Therefore, from these equations one can obtain relations coinciding in form with relations (27)–(29), but with other coefficients and functions denoted by an asterisk. Their values are given in ([5]), where the numerical method of solution is also indicated. According to this method, which is a variant of the predictor-corrector method ([7]), one of the principal stages of the computations is the solution, at each instant of time, of an equation for the integral stream function analogous to equation (29).

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CITED LITERATURE

[1] A. I. Felzenbaum, *Problems of the Theory of Wind-Driven and Thermohaline Currents*, 1968.
[2] P. S. Lineikin, *Izv. AN SSSR, Ser. Geophys.*, No. 6 (1962).
[3] V. F. Kozlov, *Izv. AN SSSR, Physics of the Atmosphere and Ocean*, **3**, No. 4, 434 (1967).
[4] A. S. Vasil' ev, *Problems of the Theory of Wind-Driven and Thermohaline Currents*, 1968.
[5] K. Bryan, M. Cox, *Tellus*, **19**, No. 1 (1967).
[6] E. N. Mikhailova, *Problems of the Theory of Wind-Driven and Thermohaline Currents*, 1968.
[7] N. N. Yanenko, *The Method of Fractional Steps in the Solution of Multidimensional Problems of Mathematical Physics*, Novosibirsk, 1966.

Note: Figure translations are in progress. See original paper for figures.