A PROPOSITION ON FUNCTIONS OF AN INTERPOLATION CLASS PROVIDING A KEY APPROACH TO THE TREATMENT OF GENERAL ANALOGUES OF THE METHOD OF SUCCESSIVE CHEBYSHEV INTERPOLATIONS

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Abstract

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A PROPOSITION ON FUNCTIONS OF AN INTERPOLATION CLASS PROVIDING A KEY APPROACH TO THE TREATMENT OF GENERAL ANALOGUES OF THE METHOD OF SUCCESSIVE CHEBYSHEV INTERPOLATIONS

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The considerations below are formulated in terms of one of the variants ((*1*), cf. also (*2–5*)) of the definition of the concept of an interpolation class, but the results will remain valid also under other known definitions (*6–8*).

Let I denote a given segment $[a, b]$ of the real axis R; let E_{n+1} denote the set of points $X = (x_0, \dots, x_n)$ of the $(n+1)$-dimensional cube I^{n+1} $(a \le x_i \le b,\ i = 0, \dots, n)$, for which the inequalities $x_0 < x_1 < \dots < x_n$ hold; let E_{n+2} denote the set of points $P = (x_0, \dots, x_{n+1})$ of the $n+2$-dimensional cube I^{n+2} $(a \le x_i \le b,\ i = 0, \dots, n+1)$ such that $x_0 < x_1 < \dots < x_{n+1}$.

A real function $\psi(x)$, defined on I, is called **alternating** on P if $(-1)^i \psi(x_i) \ge 0$ or $(-1)^i \psi(x_i) \le 0$ for $i = 0, \dots, n+1$; if, for some λ, $(-1)^i \psi(x_i) = \lambda$ $(i = 0, \dots, n+1)$, then $\psi(x)$ is called **equi-alternating** on P. We shall also speak of **strict** alternation or equi-alternation if the indicated conditions are satisfied without any of the values $\psi(x_i)$ vanishing.

$F(x; g) \equiv F(x; g_0, \dots, g_n)$ denotes a real function of $x \in I$ and of $n+1$ scalar parameters $g_0, \dots, g_n$ $(g_j \in R,\ j = 0, \dots, n)$, continuous with respect to all of its $n+2$ arguments. The definition of the function $F(x; g)$ will in what follows be regarded as given each time. In that case, when the numerical discretization of the parametric vector $g \in R^{n+1}$ is varied, the symbol $F(x; g)$ will represent a certain class Ω of continuous functions $\Phi(x) \equiv \Phi_g(x)$ $(\equiv F(x; g))$ of the argument x. This class of functions is called an **interpolation** class (of order n) if the following three conditions are fulfilled, of which the first in fact (see (*3,5*)) is already determining:

1°. However the point-sets $X = (x_0, \dots, x_n) \in E_{n+1}$ and $Y = (y_0, \dots, y_n) \in R^{n+1}$ are given, there exists in Ω one and only one function $\Phi(x) = \Phi_g(x)$ such that

$\Phi(x_i) = y_i$ $(i = 0, \dots, n)$ (the interpolation property, or J-property, of the class Ω).

2°. $\Phi(x) \in \Omega$, determined by the interpolation conditions 1°, is continuous as a function of x, X, and Y on $I \times E_{n+1} \times R^{n+1}$ (the K-property of the class Ω).

3°. There is no set $P \in E_{n+2}$ on which the difference $\Phi_{g'}(x) - \Phi_{g''}(x)$ of two distinct functions from Ω would turn out to be alternating (the A-property of the class Ω).

In what follows, by Ω we shall mean some fixed interpolation class of functions $\Phi(x) \equiv \Phi_g(x)$.

For any function $\psi(x)$, continuous on I, we take the quantity

$$\|\psi\| = \max_{x \in I} |\psi(x)|$$

as the norm of $\psi(x)$—its uniform deviation from zero on I.

It is known ([1–6]) that in Ω there exists a unique function $\Phi^*(x) = \Phi_{g^*}(x)$ for which

$$\|\Phi_{g^*}\| = \min_{\Phi \in \Omega} \|\Phi\| \equiv \rho. \tag{1}$$

Finding the vector g^* constitutes the **Chebyshev minimax problem** (more briefly, the Chebyshev problem) for the interpolation class Ω under consideration. Let us explain that, in the present exposition, as the simplest classical (rational-polynomial) case of an interpolation class Ω for which problem (1) is of interest, one should have in mind not the class of polynomials themselves $\sum g_i x^i$ $(i = 0, \dots, n)$ (for it obviously $g^* = 0$), but the "complicated" (inhomogeneous) interpolation class

$$\Phi_g(x) = \sum_{i=0}^{n} g_i x^i - f(x), \tag{2}$$

where $f(x)$ is an arbitrarily prescribed fixed function continuous on I. As is known, for $\rho > 0$ the general problem (1) has the corresponding analogues of the basic theorems of Chebyshev–Markov and Vallée-Poussin, as well as their discrete variants.

As regards the problem of constructing (with any prescribed degree of accuracy) the vector g^* that solves the Chebyshev problem (1), the variants proposed in ([1,4,8]) of analogues of the effective method ([9–11]) of successive Chebyshev interpolations (s.Ch.i.), based on certain ingenious but, in our opinion, insufficiently direct and transparent approaches to the treatment of the question of convergence, moreover leave outside consideration the substantially important question of the possible influence of small inaccuracies that are inevitably admitted here in the procedure of equalizing the moduli of $n + 2$ alternating ordinates, which for the general problem (1) is itself iterative (not completed) at each stage of the process.

If one considers the central idea of constructing a convergent process of s.Ch.i. in its original treatment ([9]), then the following presents itself as a natural basis for a comprehensive generalized treatment of it as applied to problem (1).

Theorem. Let Ω be a fixed interpolation class of order n. Let $\overline{\Phi}(x) \equiv \Phi_{\overline{g}}(x)$ denote some individually marked function of the class Ω, strictly alternating on some corresponding individually marked set $\overline{P} = (\overline{x}_0, \dots, \overline{x}_{n+1}) \in E_{n+2}$;

$$\min_{i=0,\dots,n+1} |\overline{\Phi}(\overline{x}_i)| = \overline{u}, \qquad \max_{i=0,\dots,n+1} |\overline{\Phi}(\overline{x}_i)| = \overline{v}.$$

By $\overline{\rho} = \rho(\overline{P})$ and ρ we understand, respectively, the minimal deviation from zero of functions of the class Ω on $\overline{P}$ and on I. Let us now specify that $\overline{\Phi} \in \Omega$ and $\overline{P} \in E_{n+2}$ will not here be regarded as fixed, but we shall suppose that the pair $(\overline{\Phi}, \overline{P})$ ranges over some set of marked pairs

$$M = \{(\overline{\Phi}, \overline{P})\} \subset [\Omega, E_{n+2}],$$

and let $\overline{u} \geq u_0 > 0$, where u_0 is fixed*. Finally, let $M_1 \subset M$ denote some subset of pairs $(\overline{\Phi}, \overline{P})$ subject to a condition of the form $\overline{v} - \overline{u} \geq \gamma > 0$, where γ is fixed. Then to the mentioned value γ there can be assigned such a $\delta = \delta(\gamma) >$

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* With sets of pairs of the indicated type M one has, in particular, to deal in the generalized algorithms of s.Ch.i. that are of interest to us.

> 0, that the difference $\rho - \bar{u} = \bar{\eta}$, for $(\bar{\Phi}, \bar{P}) \in M_1$, satisfies the inequality $\bar{\eta} \geq \delta$.

Proof. Let $\operatorname{sgn}\left[(-1)^i \bar{\Phi}(\bar{x}_i)\right] = v$ $(i = 0, \dots, n+1)$, where the value v, equal to $+1$ or -1, is determined by the choice of the pair $(\bar{\Phi}, \bar{P}) \in M_1$. Then, denoting by $\widetilde{\Phi}(x)$ the function of the class Ω equioscillating on $\bar{P}$, we shall have, for the same value of v,

$$(-1)^i \widetilde{\Phi}(\bar{x}_i) = v\rho = v(\bar{u} + \bar{\eta}) \qquad (i = 0, \dots, n+1). \tag{3}$$

Let i' be the value (or one of the values) of the index $i \in \{0, \dots, n+1\}$ for which

$$\left|\bar{\Phi}(\bar{x}_{i'})\right| = \bar{v}. \tag{4}$$

Put

$$X = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{i'-1}, \bar{x}_{i'+1}, \dots, \bar{x}_{n+1}) \equiv (\bar{\xi}_j)_{j=0}^{n}$$

$(\bar{X} \in E_{n+1})$. Denote by $\widehat{\Phi}(x)$ the function of the class Ω determined by the interpolation conditions

$$(-1)^i \widehat{\Phi}(\bar{x}_i) = v\bar{u} \qquad (i \in \{0, \dots, n+1\}), \quad i \neq i'. \tag{5}$$

By virtue of the K-property of the class Ω, the function

$$\Phi(x) \equiv \Phi(x; \xi_0, \dots, \xi_n; y_0, \dots, y_n),$$

defined by the interpolation conditions $\Phi(\xi_j) = y_j$ $(j = 0, \dots, n)$, where $(\xi_j)_0^n = X \in E_{n+1}$, $(y_j)_0^n = Y \in R^{n+1}$, is uniformly continuous on the $(2n+3)$-dimensional closed and bounded set $I \times E_{n+1}^{[\chi]} \times S_{n+1}$, where S_{n+1} denotes the $(n+1)$-dimensional cube $|y_j| \le \rho$, and $E_{n+1}^{[\chi]}$ is the closed subset of those systems $X = (\xi_0, \dots, \xi_n) \in E_{n+1}$ for which the continuously X-dependent separation index

$$\tau_{n+1}(X) = \min_{0 \le j \le n-1} (\xi_{j+1} - \xi_j)$$

satisfies the inequality $\tau_{n+1} \ge \chi$, with $\chi = \chi(u_0)$ denoting the minimal (certainly positive) value of the analogous continuous function

$$\tau_{n+2}(P) = \min_{0 \le i \le n} (x_{i+1} - x_i) \qquad (P \in E_{n+2})$$

on the nonempty closed (cf. [1], p. 313) subset $E_{n+2}^{(u_0)}$ of those systems $P \in E_{n+2}$ for which $\rho(P) \ge u_0$. Hence there exists a sufficiently small positive number δ, which we further subject to the condition $\delta < \gamma/3$, such that, for $\bar{\eta} < \delta$, the modulus of the difference

$$\widehat{\Phi}(\bar{x}_{i'}) - v(-1)^{i'}(\bar{u} + \bar{\eta})$$

would have to be less than $\gamma/3$. But in that case the difference $\widetilde{\Phi}(x) - \widehat{\Phi}(x)$ of two distinct functions of the class Ω would be alternating (non-strictly) on $\bar{P}$, which is impossible. Thus the existence of the required value $\delta = \delta(\gamma)$ is proved, in full agreement with the assertion of the theorem.

In the classical case of problem (2)–(1), the assertion of the theorem just established is justified in the strengthened form:

$$\bar{\eta} > \theta(\bar{v} - \bar{u}) \ge \theta\gamma \qquad (\theta = \text{const} > 0) \tag{6}$$

(where the value of θ, constant with respect to γ, depends on the given u_0); and, in view of this, for the corresponding process of s.C.i. the convergence $L_\nu = \|\Phi_\nu\|$ $(\nu \to \infty)$ to the sought minimum ρ proved to be ensured with the rate of decrease to zero of the difference $L_\nu - \rho$, at least of the type estimated by

$$L_\nu - \rho < Hq^\nu, \qquad 0 < q < 1$$

[9–11]; this fact remained valid also when rational polynomials were replaced by polynomials of any Chebyshev system of functions [10–11].

In the general case of problem (1), however, estimate (6), together with the consequence indicated from it, no longer applies. Nevertheless, as will be shown in more detail by the authors elsewhere, the theorem proved, by itself, makes it possible, by the shortest route, to establish by a uniform argument the convergence of general analogues of any of the variants [11] of the process of s.C.i. and at the same time to estab–

to establish an essential fact concerning the stability of the generalized process of s.Ch.i. with respect to admissible inaccuracies in the above-mentioned approximate alignment procedure, as required, which occur here at each of the successive stages of the process.

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Note: Figure translations are in progress. See original paper for figures.