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Abstract

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MATHEMATICS

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EXPANSION IN SOLUTIONS OF THE SCATTERING THEORY PROBLEM FOR A NON-SELF-ADJOINT SCHRÖDINGER EQUATION

(Presented by Academician L. S. Pontryagin, 23 V 1967)

The present work is devoted to the problem of expansion in eigenfunctions of the non-self-adjoint operator generated by the Schrödinger equation

$$Lu \equiv -(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2)u + q(x_1, x_2, x_3)u = \lambda^2 u \tag{1}$$

under the assumption that the complex-valued function $q(x_1, x_2, x_3) = q(x)$, $(x = (x_1, x_2, x_3))$, is defined in the whole space E_3 and satisfies the inequality

$$|q(x)| \leq C \exp\{-\varepsilon |x|\}. \tag{2}$$

1. The expansion in eigenfunctions of the operator L is given with the aid of solutions of the scattering theory problem (s.t.p.) for equation (1). (When L is self-adjoint, this was done in work ([1]).) The solution $u(x; \lambda, \omega)$ of the integral equation

$$u(x; \lambda, \omega) = e^{i\lambda(x,\omega)} - \frac{1}{4\pi} \int_{E_3} \frac{\exp\{i\lambda|x-y|\}}{|x-y|} q(y) u(y; \lambda, \omega)\, dy, \tag{3}$$

where ω is any unit vector in E_3, is called a solution of the s.t.p. for equation (1).

The following assertions hold concerning the solvability of equation (3) and concerning its solution $u(x; \lambda, \omega)$.

1°. Equation (3) is solvable for all λ for which $\operatorname{Im}\lambda > -\varepsilon/2$, with the possible exception of a countable set M with possible limit points on the line $\operatorname{Im}\lambda = -\varepsilon/2$.

2°. If equation (3) is not solvable for $\lambda = 0$, then zero is an eigenvalue of the operator L.

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3°. The solution $u(x;\lambda,\omega)$ of equation (3) is an analytic function of λ in the domain $\operatorname{Im}\lambda > -\varepsilon/2$, except at the points of the set M, at which it has poles of finite order.

4°. If $\lambda_j \in M$ and $\operatorname{Im}\lambda_j > 0$, then λ_j^2 is an eigenvalue of the operator L. Since the set $M \cap \{\lambda; \operatorname{Im}\lambda > 0\}$ is finite, the operator L has a finite number of eigenvalues $\lambda_1^2, \dots, \lambda_k^2$ with finite multiplicities $m_1, \dots, m_k$. The finiteness of the number of eigenvalues was obtained earlier in work ([2]).

If we denote $s = \lambda^2$ ($\operatorname{Im}\sqrt{s} > 0$), then

$$u(x;\sqrt{s},\omega) = \sum_{p=1}^{m_j} \frac{u_{m_j-p,j}(x)}{(s-\lambda_j^2)^p} + u_j(x;\sqrt{s},\omega), \tag{4}$$

where $u_{0,j}(x), \dots, u_{m_j-1,j}(x)$ is a chain of eigenfunctions and associated functions of the operator L corresponding to the eigenvalue λ_j^2.

5°. The operator L has no positive eigenvalues (see T. Kato ([3])). Therefore, if $\mu_j \in M$ and $\operatorname{Im}\mu_j = 0$, then μ_j^2 cannot be an eigenvalue of the operator L, and μ_j is called its spectral

singularity. The operator L has a finite number of spectral singularities $\mu_1, \dots, \mu_p$.

6°. In a neighborhood of the spectral singularity μ_j, the function $u(x;\lambda,\omega)$ is expanded in a Laurent series:

$$u(x;\lambda,\omega) = \sum_{q=1}^{n_j} \frac{v_{n_j-q,j}(x)}{(\lambda-\mu_j)^q} + v_j(x;\lambda,\omega), \tag{5}$$

where $v_j(x;\lambda,\omega)$ is regular at the point μ_j. Substituting the expansion (5) into the integral equation (3) and comparing the coefficients of equal powers of the expression $\lambda - \mu_j$, we see that

$$v_{l,j}(x) = -\frac{1}{4\pi}\int_{E_3} \frac{\exp(i\mu_j|x-y|)}{|x-y|} \left(\sum_{q=0}^{l} v_{l-q}(y)\frac{(i|x-y|)^q}{q!}\right) q(y)\,dy, \tag{6}$$

$$l = 0, \dots, n_j - 1; \qquad j = 1, \dots, p.$$

Put

$$v_{l,j}(x,\delta) = -\frac{1}{4\pi}\int_{E_3} \frac{\exp\{(i\mu_j-\delta)|x-y|\}}{|x-y|} \left(\sum_{q=0}^{l} v_{l-q,j}(y)\frac{(i|x-y|)^q}{q!}\right) q(y)\,dy. \tag{7}$$

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It is obvious that as $\delta \to 0$ the function $v_{l,j}(x-\delta) \to v_{l,j}(x)$ uniformly in every finite domain of variation of x. Suppose that the limit *

$$\lim_{\delta\to 0}\int_{E_3} v_{l_1,j}(x,\delta)v_{l_2,j}(x,\delta)\,dx = \int_{(p,u)} v_{l_1,j}(x)v_{l_2,j}(x)\,dx \tag{8}$$

exists. We shall call this limit the **regularized integral** of the functions $v_{l_1,j}(x)v_{l_2,j}(x)$.

7°. The functions $v_{0,j}(x),\dots,v_{n_j-1,j}(x)$ are linearly independent.

8°. If one puts

$$u(x;\lambda,\omega) = \sum_{q=1}^{n_j}\frac{w_{n_j-q,j}(x)}{(\lambda^2-\mu_j^2)^q} + w_j(x;\lambda,\omega) \tag{9}$$

in a neighborhood of the point μ_j, then the functions $w_{0,j}(x),\dots,w_{n_j-1,j}(x)$ are expressed through linear combinations of the functions $v_{0,j}(x),\dots,v_{n_j-1,j}(x)$, and conversely. Therefore the regularized integrals for the functions $w_{l,j}(x)$ exist and the functions $w_{0,j}(x),\dots,w_{n_j-1,j}(x)$ are linearly independent.

We shall call the system of functions $w_{0,j}(x),\dots,w_{n_j-1,j}(x)$ a **chain corresponding to the spectral singularity** μ_j.

9°. For each value of λ for which equation (3) is solvable, the asymptotic relation

$$u(x;\lambda,\omega) = e^{i\lambda(x,\omega)} + \frac{e^{i\lambda r}}{r}\left\{f(\lambda;\tilde{x},\omega) + O\left(\frac{1}{|x|^{2-h}}\right)\right\}, \tag{10}$$

holds as $|x|\to\infty$. Here $0<h<1/2$, $\tilde{x}=x/|x|$, and

$$f(\lambda;\tilde{x},\omega) = -\frac{1}{4\pi}\int_{E_3} e^{-i\lambda(\tilde{x},y)}q(y)u(y;\lambda,\omega)\,dy. \tag{11}$$

10°. For real values of λ with $|\lambda| \geq \max_j\{|\mu_j|+1\}$, the function $u(x;\lambda,\omega)$ is uniformly bounded in all arguments.

2. Denote by $R(x;y;\lambda)$ the kernel of the resolvent of equation (1). Then it is obvious that

$$R(x;y;\lambda) = \frac{1}{4\pi}\frac{\exp(i\lambda|x-y|)}{|x-y|} - \frac{1}{4\pi}\int_{E_3}\frac{\exp(i\lambda|x-z|)}{|x-z|}q(z)R(z;y;\lambda)\,dy. \tag{12}$$

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* Here it should be noted that as $|x| \to \infty$ the functions $v_{l,j}(x)$ grow, but not faster than $|x|^{l-1}$.

It is not difficult to prove that $R(x; y; \lambda)$ is an analytic function of λ for $\operatorname{Im}\lambda > -\varepsilon/2$, with the exception of the set M. In neighborhoods of points of the discrete spectrum, a method is known (see, for example, ([4,5])) for determining the principal parts of the resolvent kernel. Therefore we shall not dwell on this, and shall denote by $R_j(x; y; \lambda)$ the principal part of the resolvent in a neighborhood of the point λ_j^2. We must also be able to determine the principal parts $A_j(x; y; \lambda)$ of the resolvent in a neighborhood of the spectral singularities μ_j $(j = 1, \dots, p)$ in terms of the principal parts $w_{0,j}(x), \dots, w_{n_j-1,j}(x)$ of the solution of the b.v.p. Without giving detailed arguments, we note that the regularized integrals introduced above make it possible to find $A_j(x; y; \lambda)$ in terms of $w_{0,j}(x), \dots, w_{n_j-1,j}(x)$ by the formulas*

$$A_j(x; y; \lambda) = \sum_{q=1}^{n_j} \frac{\alpha_{q,j}(x; y)}{(\lambda^2 - \mu_j^2)^q}, \tag{13}$$

$$\alpha_{q,j}(x; y) = \sum_{s=1}^{n_j-q} C_{s+q,j} \sum_{\nu=0}^{s} w_{\nu,j}(x) w_{s-\nu,j}(y), \tag{14}$$

$$C_{n_j,j}\alpha_{0,n_j-1}^{(j)} = -1, \qquad C_{n_j-1,j} = \alpha_{1,n_j-1}^{(j)} / \left(\alpha_{0,n_j-1}^{(j)}\right)^2, \tag{15}$$

$$C_{n_j-l+1,j} = (-1)^l \left(\alpha_{0,n_j-1}^{(j)}\right)^{-k} \begin{vmatrix} \alpha_{1,n_j-1}^{(j)} & \alpha_{0,n_j-1}^{(j)} & 0, \dots, & 0 \\ \alpha_{2,n_j-1}^{(j)} & \alpha_{1,n_j-1}^{(j)} & \alpha_{0,n_j-1}^{(j)} & 0, \dots, 0 \\ \vdots & \vdots & \vdots & \end{vmatrix}, \tag{16}$$

where

$$\alpha_{kl}^{(j)} = \int_{(p,\omega)} w_{kj}(x) w_{lj}(x)\, dx. \tag{17}$$

3. We now formulate the main result of the paper. First we introduce some notation. Among the spectral singularities of equation (1) there may be some symmetric with respect to zero. Suppose that the numbers $\mu_1, \dots, \mu_p$ are numbered so that $\mu_1 = -\mu_p, \dots, \mu_{p_1} = \mu_{p-p_1+1}$, and $\mu_1, \dots, \mu_{p_1} < 0$. Denote by Γ the curve composed of subsegments of the half-axis $\lambda \geq 0$ and small semicircles around the poles of the functions $u(x; \lambda, \omega)$ and $u(x; -\lambda, \omega)$ on the half-axis $\lambda > 0$. The semicircle lies in the upper half-plane if its center is a pole of the function $u(x; \lambda, \omega)$, and in the lower half-plane if its center is a pole of the function $u(x; -\lambda, \omega)$ and this pole does not coincide with any pole of the function $u(x; \lambda, \omega)$.

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Theorem. *Let the function $f(x)$ vanish outside a certain compact set, have continuous first-order derivatives, and let $Lf \in L_2(E_3)$. Let $R(x;y;\lambda)$ be the resolvent kernel of equation (1) and satisfy condition (2). Then the following assertions hold:*

a) *$R(x;y;\lambda)$ is an analytic function of λ in the half-plane $\operatorname{Im}\lambda > -\varepsilon/2$, with the exception of a countable set M with possible limit points on the line $\operatorname{Im}\lambda = -\varepsilon/2$;*

b) *$R(x;y;\lambda)$ is a kernel of Carleman type if and only if $\operatorname{Im}\lambda > 0$ and λ is distinct from $\lambda_1, \dots, \lambda_k$;*

c) *if $\operatorname{Im}\lambda > 0$ and λ is distinct from $\lambda_1, \dots, \lambda_k$, then*

$$\int_{E_3} R(x;y;\lambda) f(y)\, dy = \sum_{j=1}^{k} \int_{E_3} R_j(x;y;\lambda) f(y)\, dy +$$

$$+ \sum_{j=1}^{p_1} \int_{E_3} A_j(x;y;\lambda) f(y)\, dy + \frac{1}{(2\pi)^3} \int_{\Gamma} \int_{W_s} \frac{u(x;-s,\omega)}{s^2-\lambda^2} F(s,\omega) s^2\, ds\, d\omega. \quad (18)$$

* For simplicity we assume that each spectral singularity is assigned only one chain. Otherwise $A_j(x;y;\lambda)$ equ

Here

$$F(s,\omega) = \int_{E_3} f(x) u(x;s,\omega)\, dx; \quad (19)$$

W_3 is the unit sphere in E_3, and $d\omega$ is the elementary measure on W_3;

d) the last integral in formula (18) converges uniformly with respect to the variable x from E_3.

We shall make several remarks on the method of proof of assertion c), since the remaining assertions are obvious. Formula (18) is in fact an expansion in the solutions of the scattering-theory problem for equation (1). Expanding the solution of the scattering-theory problem in spherical functions, our problem can be reduced to the problem of expansion in the solutions of the scattering-theory problem for an infinite system of ordinary differential equations with singularities of the form $l(l+1)/x^2$, $l = 0, 1, \dots$, which grow rapidly as l increases. Next we first obtain the expansion formula in the solutions of the scattering-theory problem for a finite system of differential equations with singularities, and then carry out the limiting passage from the finite system to the infinite one, which leads us to the goal.

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Let us note that if the function $f(x)$ is finite and lies in the domain of definition of the operator L^2, then the operator $L - \lambda^2$ can be applied to formula (18), and from it one obtains an expansion of the function $f(x)$ in the solutions of the scattering-theory problem for equation (1).

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REFERENCES

1. A. Ya. Povzner, DAN, 104, No. 3, 360 (1955).

2. R. M. Martirosyan, Izv. AN SSSR, Ser. Mat., 24, No. 6, 897 (1960).

3. T. Kato, Comm. on Pure and Appl. Math., 12, No. 3, 403 (1959).

4. E. Kamke, *Handbook of Ordinary Differential Equations*, IL, 1951.

5. M. A. Naimark, Tr. Moscow Math. Soc., 3, 181 (1954).

Note: Figure translations are in progress. See original paper for figures.

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