INITIAL THERMOSTRUCTURAL STRESSES

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THEORY OF ELASTICITY

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INITIAL THERMOSTRUCTURAL STRESSES IN TWO-COMPONENT MEDIA

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A statistical boundary-value problem of thermoelasticity has been solved for a macroscopically homogeneous quasi-isotropic two-component medium with isotropic components, whose properties are described by moment functions up to infinite order inclusive. Closed-form formulas are obtained for the mean values and variances of stresses in the components of a medium uniformly cooled from some temperature of the natural state in the absence of external forces.

§ 1. Let $\mathbf{C}^{\mathrm{I}}, \mathbf{C}^{\mathrm{II}}, \mathbf{a}^{\mathrm{I}}, \mathbf{a}^{\mathrm{II}}$ be deterministic tensors of the elastic moduli and coefficients of linear expansion of the I and II components of an elastic medium; $\lambda(\mathbf{x})$ a random function describing the distribution of the components: $\lambda(\mathbf{x}) = 1$, if $M \in L^{\mathrm{I}}$, and $\lambda(\mathbf{x}) = 0$, if $M \in L^{\mathrm{II}}$; L^{I} and L^{II} are subsets of points belonging to the I and II components; $L^{\mathrm{I}} \cup L^{\mathrm{II}} = L$ is the set of corresponding points $M(\mathbf{x})$. Then the physical properties of the medium are described by the random functions

$$\theta(\mathbf{x}) = \mathbf{C}^{\mathrm{I}}\lambda(\mathbf{x}) + \mathbf{C}^{\mathrm{II}}[1 - \lambda(\mathbf{x})]; \qquad \alpha(\mathbf{x}) = \mathbf{a}^{\mathrm{I}}\lambda(\mathbf{x}) + \mathbf{a}^{\mathrm{II}}[1 - \lambda(\mathbf{x})]. \tag{1}$$

If the medium is macroscopically homogeneous, then the moment functions describing the structure are, in the limit, representable by the expressions:

$$K_\lambda^{(m)}(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^m) \equiv \langle \lambda_1^0 \lambda_2^0 \dots \lambda_m^0 \rangle = D_\lambda^{(m)} \prod_{n=1}^{m-1} k(|\mathbf{x}^{n-1}\mathbf{x}^n|), \tag{2}$$

where $\lambda_k^0 = \lambda^0(\mathbf{x}^k) = \lambda - P$; $P = \langle \lambda \rangle = \text{const}$ with respect to $\mathbf{x}$; $\langle \dots \rangle$ denotes the sign of mathematical expectation; $m \geqslant 2$; $k(x) = 1,\ x = 0$; $k(x) = 0,\ x \neq 0$; $D_\lambda^{(m)} = P(1-P)^m + (1-P)(-P)^m$.

Let $\xi(\mathbf{x})$, $\varepsilon(\mathbf{x})$, $\chi(\mathbf{x})$ be random tensors of stresses, strains, and displacements. Suppose the properties of the medium are described by the equations:

$$\operatorname{div}\xi = 0; \qquad \varepsilon = \operatorname{def}\chi; \qquad \xi = \theta(\varepsilon - \alpha T), \tag{3}$$

where T is a deterministic temperature difference ($T = \text{const}$ with respect to $\mathbf{x}$). If there are no external forces, then $\mathbf{p} = \langle\xi\rangle = 0$; $\xi^0 = \xi - \mathbf{p} = 0$ on the boundary S of the domain V; $\mathbf{e} = \langle\varepsilon\rangle = \mathbf{a}^0 T$; $\mathbf{a}^0$ is the coefficient of linear expansion of the two-component medium, determined below through the physical properties of the components.

In the case where the deformations $\varepsilon^T = \alpha T$ are incompatible, stresses ξ^{I} and ξ^{II} appear in the components of the medium. It is required to determine: $\mathbf{p}^{\mathrm{I}} = \langle\xi^{\mathrm{I}}\rangle$; $\mathbf{p}^{\mathrm{II}} = \langle\xi^{\mathrm{II}}\rangle$; $D^{\mathrm{I}} = \langle(\xi^{0\mathrm{I}})^2\rangle$; $D^{\mathrm{II}} = \langle(\xi^{0\mathrm{II}})^2\rangle$, where $\xi^{0\mathrm{I}} = \xi^{\mathrm{I}} - \mathbf{p}^{\mathrm{I}}$, $\xi^{0\mathrm{II}} = \xi^{\mathrm{II}} - \mathbf{p}^{\mathrm{II}}$. These stresses are "initial" at the temperature T_1 at which the body is loaded by external forces.

§ 2. The solution of system (3) under the boundary conditions $\xi^0 \cdot \mathbf{n} = 0$ ($\mathbf{n}$ is the unit outward normal to the boundary S) is reduced to the solution of a system of integral-

equations:

$$\varepsilon^0(\mathbf{x}) = \int\limits_{\dot{V}} \nabla\mathbf{G}(\mathbf{x}, \mathbf{x}') \cdot d \cdot \mathbf{v}\, \Pi(\mathbf{x}')\, dV'. \tag{4}$$

Here $\varepsilon^0 = \varepsilon - \mathbf{e}$; $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ is the Green tensor; $\Pi = \Delta e - \chi T + \Delta\varepsilon^0$; $\Delta = \theta - \mathbf{C}$; $\chi = \beta - \mathbf{B}$; $\beta = \theta\alpha$; $\mathbf{C} = \langle\theta\rangle$; $\mathbf{B} = \langle\beta\rangle$. The system (4) is solved by the method of iterations. Below is given the solution of (4) for the case when both components are isotropic and the medium is isotropic on the average:

$$\varepsilon^0 = fT \sum_{k=1}^{\infty} (3K^*)^{k-1} \sum_{m=1}^{k} \mathbf{I}^{(m)} \lambda_m^0, \tag{5}$$

where $f = 3(K^* a^0 - n^*)$; $K^* = K^{\mathrm{I}} - K^{\mathrm{II}}$; $K^{\mathrm{I}}, K^{\mathrm{II}}$ are the bulk-compression moduli of components I and II; $n^* = K^{\mathrm{I}} a^{\mathrm{I}} - K^{\mathrm{II}} a^{\mathrm{II}}$; $a^0_{ij} = a^0 \delta_{ij}$; the tensor $\mathbf{I}^{(m)} \lambda_m^0$ has components

$$I_{ij}^{(m)} \lambda_m^0 = \int\limits_{\dot{V}} \frac{\partial G_{i\varphi}(\mathbf{x}^{m-1}, \mathbf{x}^m)}{\partial x_j^{(m-1)}} \frac{\partial \lambda_m^0}{\partial x_\varphi^{(m)}}\, dV_m.$$

In consequence of the properties (2), for a region V of finite dimensions the tensor $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ may be replaced by the Somigliana tensor (the regular part of the tensor is of no essential importance). Taking (5) and the results of work (^{1}) into account, we obtain formulas for the central moments of second order of the random stresses:

$$D_{ijmn} \equiv \langle \varepsilon^0_{ij} \varepsilon^0_{mn} \rangle = m^2 T^2 D^{(2)}_\lambda \left[R\delta_{ij}\delta_{mn} + Q(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \right], \tag{6}$$

where

$$R = \frac{8}{5}d^2 + \left(K^2 q^2 + 2KK^* q - \frac{8}{15} d^2 q^2 + \frac{6KK^* a^* q}{m} \right) D^{(2)}_\lambda;$$

$$Q = \frac{4}{15} d^2 (1 + q^2 D^{(2)}_\lambda); \qquad m = \frac{fg}{c+2d}; \qquad q = -\frac{K^*}{c+2d};$$

$$K = c + \frac{2}{3}d; \qquad g = (1+qP)^{-1}[1 - q(1-P)]^{-1}; \qquad a^* = a^{\mathrm{I}} - a^{\mathrm{II}};$$

c, d are the mean values of the Lamé coefficients. The quantity a^0 is determined by averaging the physical equations of the system (3), solved with respect to the strains, taking (5) into account,

$$a^0 = (Ka + K^* a^* D^{(2)}_\lambda - n^* qg D^{(2)}_\lambda)/(K + K^* qg D^{(2)}_\lambda); \qquad a = \langle \alpha \rangle.$$

Another method for determining the coefficient a^0 was proposed by V. V. Bolotin and V. N. Moskalenko (^{2}).

The stresses at the point M of the region V

$$\xi(\mathbf{x}) = \xi^{\mathrm{I}} \lambda(\mathbf{x}) + \xi^{\mathrm{II}}[1 - \lambda(\mathbf{x})] \tag{7}$$

are defined on the set of realizations L; the stresses ξ^{I} and ξ^{II} are defined on L^{I} and L^{II}, respectively. In order that the averaging operator over the set L could be applied to formula (7), we introduce the functions η^{I} and η^{II}:

$$\xi(\mathbf{x}) = \eta^{\mathrm{I}} \lambda(\mathbf{x}) + \eta^{\mathrm{II}}[1 - \lambda(\mathbf{x})]; \tag{8}$$

$$\eta^{\mathrm{I}} = \begin{cases} \xi^{\mathrm{I}}, & L^{\mathrm{I}}, \\ p^{\mathrm{I}}, & L^{\mathrm{II}}, \end{cases} \qquad \eta^{\mathrm{II}} = \begin{cases} \xi^{\mathrm{II}}, & L^{\mathrm{II}}, \\ p^{\mathrm{II}}, & L^{\mathrm{I}}. \end{cases}$$

Multiplying both sides of (8) by λ and applying the mathematical expectation operator, we find:

$$p^{\mathrm{I}}_{ij} = p_{ij} + P^{-1} \langle \lambda^0 \xi^0_{ij} \rangle; \qquad p^{\mathrm{II}}_{ij} = p_{ij} - (1-P)^{-1} \langle \lambda^0 \xi^0_{ij} \rangle, \tag{9}$$

where, by the condition of the problem, $P_{ij} = 0$.

From the relation $\langle \lambda \xi_{ij}^2 \rangle = \langle \lambda (\eta_{ij}^I)^2 \rangle$, obtained on the basis of (8), after introducing variations and elementary transformations we have

$$\begin{aligned} D_{ijij}^I &= D_{ijij} + P^{-1} \langle \lambda^0 (\xi_{ij}^0)^2 \rangle - (p_{ij}^I)^2, \\ D_{ijij}^{II} &= D_{ijij} - (1-P)^{-1} \langle \lambda^0 (\xi_{ij}^0)^2 \rangle - (p_{ij}^{II})^2. \end{aligned} \tag{10}$$

Calculations by formulas (9) give

$$\begin{aligned} p_{ij}^I &= D_\lambda^{(2)} P^{-1} T \delta_{ij} (f - mK^* h - Km); \\ p_{ij}^{II} &= -P(1-P)^{-1} p_{ij}^I; \qquad h = 1 - 2P + qD_\lambda^{(2)}. \end{aligned} \tag{11}$$

The moments in formulas (10) are determined analogously:

$$\langle \lambda^0 (\xi_{ij}^0)^2 \rangle = D_\lambda^{(2)} T^2 \delta_{ij} \left\{ (1-2P) f^2 g^2 - 2Kfmh + m^2 \left[K^2 (h + qD_\lambda^{(2)}) + 2KK^* h^2 \right] \right\}. \tag{12}$$

Formulas (5), (11), and (12) are valid if the corresponding series converge. The convergence condition is: $|q(1-P)| < 1$.

It follows from formulas (5) and (10)–(12) that the variances of the initial thermostructural stresses are nonzero, as are the mean values of the normal stresses in the components. The mean values of the shear stresses in the components are equal to zero for nonzero variances.

Numerical calculations based on initial data characteristic of glass plastics show that, in the region $0.3 \le P \le 0.7$, the coefficients of variation of the initial thermostructural stresses in the components are not small in comparison with unity and are of the same order as the coefficients of variation of the physical properties.

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1. S. D. Volkov, V. Ya. Dolgikh, *Mekhanika polimerov*, No. 3 (1967).
2. V. V. Bolotin, V. N. Moskalenko, DAN, 178, No. 3, 563 (1968).