THEORY OF ELASTICITY

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Abstract

Full Text

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ON A THEORY OF BENDING OF ANISOTROPIC THREE-LAYER ELASTIC PLATES WITH A CORE

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1. Theories of three-layer shells and plates with a core have recently received considerable attention ([1−3]). Nevertheless, the theory of anisotropic three-layer plates has not yet received sufficient development.

In what follows, a theory of bending of symmetric three-layer elastic plates will be justified separately for the cases of a light and of a stiff core, and the matching conditions on the boundary surfaces will be strictly satisfied.

2. The case of three-layer plates with a stiff core. Let us have a three-layer plate of constant thickness $2\bar{h}+h'+h''$, $h'=h''$ being the thickness of the outer layers; $2\bar{h}$ the thickness of the middle layer. Here and below, quantities characterizing the upper and lower layers and the middle layer will be denoted respectively by primes $'$, double primes $''$, and by an overbar. It is assumed that the material of the outer layers possesses identical physicomechanical characteristics. The middle surface of the plate (coinciding with the middle surface of the core) is referred to an arbitrary curvilinear coordinate system x^i (Greek indices everywhere take the values 1, 2, and Roman indices the values 1, 2, 3); the coordinate $x^3 = z$ ($|z| \le \bar{h}+h'$) is normal to the middle surface of the plate.

The following hypotheses are adopted:

A. The layers of the plate are regarded as anisotropic, having at each point an elastic plane of symmetry parallel to the middle surface of the plate.

B. For the outer layers, the Love–Kirchhoff hypothesis is assumed to be valid.

C. For the middle layer, the theory of plates of moderate thickness is assumed to be valid (see ([4])), to which we add the hypothesis of incompressibility of the core in the normal direction ($\bar{e}_{33}=0$).

On the basis of these hypotheses (see also ([5])) we take

$$\bar{\tau}^{\omega 3} = \bar{\varphi}^{\omega}(z^2-\bar{h}^2)+f^{\omega}, \tag{1}$$

where $\bar{\varphi}^{\omega} = \bar{\varphi}^{\omega}(x^{\lambda})$, $f^{\omega} = f^{\omega}(x^{\lambda})$ are as yet unknown functions, whose meaning is clear.

On the basis of hypothesis B we obtain expressions for the tangential displacements in the outer layers

$$\begin{aligned} V'_{\alpha} &= U_{\alpha} + (A - z)V'_3|_{\alpha} \quad \text{for } \bar{h} + h' \geq z \geq \bar{h}, \\ V''_{\alpha} &= -U_{\alpha} - (A + z)V''_3|_{\alpha} \quad \text{for } -\bar{h} - h' \leq z \leq -\bar{h}, \end{aligned} \tag{2}$$

where $A = \bar{h} + h'/2$; U_{α} are the tangential displacements in the middle surface of the upper layer.

From the equation

$$\bar{V}_{\alpha,3} = 4\bar{F}_{\alpha 3\lambda 3}\bar{\tau}^{\lambda 3} - \bar{V}_3|_{\alpha}, \tag{3}$$

obtained from the physical equation

$$\bar{e}_{\lambda 3} = 2\bar{F}_{\alpha 3\lambda 3}\bar{\tau}^{\omega 3} \tag{4}$$

and the geometrical equation

$$\bar{e}_{ij} = \frac{1}{2}(\bar{V}_i|j + \bar{V}_j|i) \tag{5}$$

(the comma and the index i following it denote partial differentiation with respect to the coordinate x^i; the vertical bar denotes covariant differentiation; E^{ijmn} and F_{ijmn} are the tensors of elastic moduli of the material of the layers ([6])); taking into account that in the present case $\bar{V}_3 = V'_3 = V_3$, from (3)–(5) we obtain

$$\bar{V}_{\alpha} = -zV_{3|\alpha} + 4z\bar{F}_{\alpha 3\lambda 3}(f^{\lambda} - \bar{h}^2\bar{\varphi}^{\lambda}) + \frac{4}{3}z^3\bar{F}_{\alpha 3\lambda 3}\bar{\varphi}^{\lambda} \quad \text{for } |z| \leq \bar{h}. \tag{6}$$

From the exact equilibrium equation

$$\bar{\tau}^{\omega 3}|_{\omega} + \bar{\tau}^{33}{}_{,3} = 0 \tag{7}$$

taking (1) into account, we derive

$$\bar{\tau}^{33} = -\frac{1}{3}\varphi^{\alpha}{}_{|\alpha}z^3 + (\bar{h}^2\varphi^{\alpha}{}_{|\alpha} - f^{\alpha}{}_{|\alpha})z. \tag{8}$$

At $z = \pm(h + h')$ it satisfies the conditions

$$
{}'\tau^{\alpha 3} = {}''\tau^{\alpha 3} = 0; \qquad {}'\tau^{33} = \pm\frac{1}{2}p_{(0)}, \tag{9}
$$

where $p_{(0)} = p_{(0)}(x^\omega)$ is the external normal load. According to hypothesis B,

$$
{}'\tau^{\omega\pi} = {}'\widetilde{E}^{\omega\pi\sigma\rho}\, {}'e_{\sigma\rho}, \tag{10}
$$

where ${}'e_{\sigma\rho}$ is the tensor of tangential deformation corresponding to the upper layer,

$$
{}'\widetilde{E}^{\omega\pi\sigma\rho} = {}'E^{\omega\pi\sigma\rho} - \frac{{}'E^{\omega\pi 33}\, {}'E^{33\sigma\rho}}{{}'E^{3333}}. \tag{11}
$$

Applying relations (10), (9), (5), (2), from the exact equilibrium equations we derive

$$
{}'\tau^{\beta 3} = \frac{1}{2}\, {}'\widetilde{E}^{\alpha\beta\rho\theta} \Big[(U_{\rho|\alpha} + U_{\theta|\rho\alpha})(\bar{h} + h' - z) +
$$

$$
+ \left\langle 2A(\bar{h} + h' - z) - (\bar{h} + h')^2 + z^2 \right\rangle V'_{3\,|\rho\theta\alpha} \Big], \tag{12}
$$

$$
{}'\tau^{33} = \frac{1}{2}\, {}'\widetilde{E}^{\alpha\beta\rho\theta} \left[(U_{\rho|\alpha\beta} + U_{\theta|\rho\alpha\beta}) \left\langle \frac{1}{2}(\bar{h} + h')^2 - z(\bar{h} + h') + \frac{1}{2}z^2 \right\rangle + \right.
$$

$$
\left. + \left\langle \frac{1}{3}(\bar{h} + h')^2 \left(\bar{h} - \frac{1}{2}h' \right) - zh(\bar{h} + h') + z^2 A - \frac{1}{3}z^3 \right\rangle V'_{3\,|\rho\theta\alpha\beta} \right] + \frac{1}{2}p_{(0)}.
$$

Imposing the geometrical and static matching conditions

$$
\bar{V}_\alpha|_{z=\bar{h}-0} = V'_\alpha|_{z=\bar{h}+0}, \qquad {}'\tau^{\beta 3}|_{z=\bar{h}+0} = \bar{\tau}^{\beta 3}|_{z=\bar{h}-0}, \tag{13}
$$

we derive, respectively,

$$
\bar{\varphi}^\omega = \frac{3}{2\bar{h}^3} \left[\bar{h} f^\omega - \bar{E}^{\alpha 3\omega 3}(U_\alpha + AV_{3|\alpha}) \right], \qquad f^\rho = \frac{h'}{2}\, {}'\widetilde{E}^{\lambda\rho\omega\pi}(U_{\omega|\pi\alpha} + U_{\pi|\omega\alpha}). \tag{14}
$$

Knowing that the tensors of shearing forces and moments are determined by the relations

$$N_{(0)}^{\omega} = 2\int_0^{\bar h} \bar\tau^{\omega 3}\,dz + 2\int_{\bar h}^{\bar h + h'} {}'\tau^{\omega 3}\,dz, \qquad L_{(1)}^{\omega\rho} = 2\int_0^{\bar h} \bar\tau^{\omega\rho} z\,dz + 2\int_{\bar h}^{\bar h + h'} {}'\tau^{\omega\rho} z\,dz, \tag{15}$$

where

$$\bar\tau^{\omega\rho} = \widetilde{\bar E}^{\omega\rho\sigma\pi}\bar e_{\sigma\pi} + \frac{\bar E^{\omega\rho 33}}{\bar E^{3333}}\bar\tau^{33}, \tag{16}$$

and taking (1), (8), (10), (12), (15) into account in the equilibrium equations

$$N^{\alpha}_{(0)|\alpha} + p_{(0)} = 0, \qquad L^{\omega\rho}_{(1)|\omega} - N^{\rho}_{(0)} = 0, \tag{17}$$

we derive the resolving system of equations in the principal unknowns U_α and V_3

$$\frac{1}{2}{h'}^{2}\,{}'\widetilde{E}^{\alpha\beta\rho\theta}(U_{\rho|\theta\alpha\beta} + U_{\theta|\rho\alpha\beta}) - \frac{1}{6}{h'}^{3}\,{}'\widetilde{E}^{\alpha\beta\rho\theta}V_{3|\rho\theta\alpha\beta} + 2\bar E^{\alpha 3\beta 3}(U_{\omega|\beta} + AV_{3|\alpha\beta}) + p_{(0)} = 0,$$

$$\begin{gathered}{}^{2}/_{3}\bar h^{3}\bar E^{\mu\lambda\omega\pi}V_{3|\omega\pi} - 2\bar E^{\alpha 3\lambda 3}(U_{\alpha} + AV_{3|\alpha}) + \\ + {}^{8}/_{5}\bar h^{2}\bar E^{\alpha 3\beta 3}[(U_{\alpha|\pi\mu} + AV_{3|\alpha\pi\mu})\bar F_{\omega 3\beta 3} + (U_{\alpha|\omega\mu} + AV_{3|\alpha\omega\mu})\bar F_{\pi 3\beta 3}]\bar E^{\mu\lambda\omega\pi} - \\ - {}^{4}/_{5}\bar h^{2}\frac{\bar E^{\alpha 3\beta 3}\bar E^{\mu\lambda 33}}{\bar E^{3333}}(U_{\alpha|\beta\mu} + AV_{3|\alpha\beta\mu}) + \bar h h'\bar E^{\mu\lambda\omega\pi}(U_{\omega|\pi\mu} + U_{\pi|\omega\mu}) = 0.\end{gathered} \tag{18}$$

Let us note that the first equation (18) can also be obtained from the conditions

$$\bar\tau^{33}\big|_{z=\bar h - 0} = {}'\tau^{33}\big|_{z=\bar h + 0}. \tag{19}$$

The system (18) is a generalization of the system obtained in (5). The equations corresponding to the case of a transversely isotropic three-layer plate are obtained by taking for E^{ijkl} and F_{ijmn} the expressions given in (4). Expressing U_α from the relation

$$2\bar G_3 U^{\omega} = DV_{3|\lambda}^{\;\omega\lambda} - 2\bar G_3 AV_3^{\;|\omega} + \varepsilon^{\omega\lambda}\Phi_{|\lambda} - \left\langle \frac{2}{5}\frac{\bar h^2}{1-\bar\nu}\left(2\frac{\bar G}{\bar G_3} - \bar\nu_3\frac{\bar E}{\bar E_3}\right) + \frac{E'\bar h h'}{\bar G_3(1-{\nu'}^2)}\right\rangle p_{(0)}^{\;|\omega} \tag{20}$$

$$(\varepsilon^{11} = \varepsilon^{22} = 0; \quad \varepsilon^{12} = -\varepsilon^{21} = 1),$$

we reduce equations (18) to the equations

$$DV_{3|\omega\lambda}^{\ |\omega\lambda} = p_{(0)} - \left\{ \frac{2}{5} \frac{\bar{h}^2}{1-\bar{\nu}} \left(2\frac{\bar{G}}{\bar{G}_3} - \bar{\nu}_3 \frac{\bar{E}}{\bar{E}_3} \right) + \frac{E'\bar{h}h'}{\bar{G}_3(1-\nu'^2)} \right\} p_{(0)|\omega}^{\ |\omega}, \tag{21}$$

$$\frac{1}{\bar{G}_3} \left(\frac{2}{5}\bar{h}^2\bar{G} + G'\bar{h}h' \right) \Phi_{|\lambda\rho}^{\ |\lambda\rho} - \Phi_{|\sigma}^{\ |\sigma} = 0,$$

where

$$D = \frac{2}{3} \left[\frac{\bar{E}\bar{h}^3}{1-\bar{\nu}^2} + \frac{((\bar{h}+h')^3 - \bar{h}^3)E'}{1-\nu'^2} \right];$$

E', ν' are the modulus of elasticity and Poisson' s ratio of the material of the outer layers, and $\bar{E}$, $\bar{\nu}$ and $\bar{E}_3$, $\bar{\nu}_3$ are those of the filler material in the plane, respectively parallel and perpendicular to the middle plane of the filler. Let us note that in the case $\bar{h} \to 0$ one obtains the equations of plate bending within the framework of the classical theory, corresponding to the material of the outer layers; in the case $h' \to 0$, the equations of bending of homogeneous plates with the filler material, these equations being a generalization to the transversely isotropic case of the equations obtained in (6) in the isotropic case by applying the algorithm given in (7).

3. The case of three-layer plates with a light filler. Suppose that hypotheses A and B are valid. For the middle layer, the theory of plates of mean thickness will be supplemented by the hypothesis $\bar{\tau}^{\omega\pi} = 0$. We shall also assume that through the thickness

$$\bar{V}_3 = \overset{(0)}{\bar{V}}_3 + z^2 \overset{(2)}{\bar{V}}_3. \tag{22}$$

From the exact equilibrium equations we obtain

$$\bar{\tau}^{\omega 3} = \bar{g}^{\omega}, \qquad \bar{\tau}^{33} = -z\bar{g}^{\omega}{}_{|\omega}, \tag{23}$$

where $\bar{g}^{\omega} = \bar{g}^{\omega}(x^{\lambda})$.

The displacement components $\bar{V}_{\alpha}$ are given by the relation

$$\bar{V}_{\alpha} = -\overset{(0)}{\bar{V}}_{3|\alpha} - 4\bar{F}_{\alpha\lambda3\beta}\bar{g}^{\lambda}z - \delta_{\perp}\, {}^1/_3 z^3 \overset{(2)}{\bar{V}}_{3|\alpha}, \tag{24}$$

where $\delta_{\perp}$ identifies the effect of compressibility of the filler in the normal direction ($\delta_{\perp} = 1$ in the case when this effect is taken into account; $\delta_{\perp} = 0$ in the opposite case, $\bar{e}_{33} = 0$).

Knowing that $\overset{(2)}{\bar{V}}_3$ is determined by relation (4)

$$\overset{(2)}{\bar{V}}_3 = \frac{1}{2\bar{E}^{3333}}\left(\bar{\tau}^{33} - \bar{E}^{33\alpha\beta}\bar{e}_{\alpha\beta}\right), \tag{25}$$

and, taking into account (23), (24), (5), from the geometric matching condition $\overline{V}_3\big|_{z=\bar{h}-0} = V_3'\big|_{z=\bar{h}+0}$, we obtain

$$V_3' = \overline{V}_3^{(0)} - \frac{\bar{h}^2}{2\bar{E}^{3333}}\left\{\bar{g}^{\alpha\gamma}{}_{|\alpha} + \bar{E}^{33\alpha\beta}\left\langle 2(\bar{E}_{\alpha\lambda3}\bar{g}^{\lambda})_{|\beta} + \bar{F}_{\beta\lambda3}\bar{g}^{\lambda}{}_{|\alpha}\right\rangle - \overline{V}_3^{(0)}{}_{|\alpha\beta}\right\}. \tag{26}$$

Condition (13,2) leads for $\bar{g}^{\beta}$ to an expression identical with that given by relation (14,2), while from condition (13,1), with account of (24) and (2,1), we obtain

$$\left(U_\alpha + A\overline{V}_3^{(0)}{}_{|\alpha}\right)\bar{E}^{\alpha3\gamma3} = \frac{1}{2}hh'\,\widetilde{E}^{\omega\gamma\rho\theta}\left(V_{\rho|\theta\omega} + U_{\theta|\rho\omega}\right) - \delta_1\frac{\bar{h}^2}{2}\left(\frac{\bar{h}}{3} + \frac{h'}{2}\right)\frac{\bar{E}^{33\rho\pi}\bar{E}^{\rho3\gamma3}}{\bar{E}^{3333}}\overline{V}_3^{(0)}{}_{|\rho\pi\alpha}. \tag{27}$$

This is one of the resolving equations of the problem, to which must be added the equation

$$'\widetilde{E}^{\alpha\beta\rho\theta}\left(U_{\rho|\theta\alpha\beta} + U_{\theta|\rho\alpha\beta}\right)\frac{h'^2}{2} + 2\bar{E}^{\alpha3\gamma3}\left(U_{\alpha|\gamma} + A\overline{V}_3^{(0)}{}_{|\alpha\gamma}\right) -$$

$$-\frac{h'^3}{6}\,\widetilde{E}^{\alpha\beta\rho\theta}\overline{V}_3^{(0)}{}_{|\rho\theta\alpha\beta} = \delta_1 h^2\left(\frac{\bar{h}}{3} + \frac{h'}{2}\right)\frac{\bar{E}^{\alpha3\gamma3}\bar{E}^{33\rho\pi}}{\bar{E}^{3333}}\overline{V}_3^{(0)}{}_{|\rho\pi\alpha\gamma} + p_{(0)} = 0, \tag{28}$$

which is obtained either from (17,1), (15), (2), (12,1), (26), or from condition (19).

The resolving system of equations consists of equations (27,1) in the basic unknowns $\overline{V}_3^{(0)}$ and U_α (from equation (17,2) an equation is obtained which is identically satisfied by equation (27)).

It is noted that in the case $\delta_1 = 0$ ($\bar{e}_{33} = 0$), the equations corresponding to three-layer plates with a light core are obtained for the case with a rigid core by taking in the latter $\bar{E}^{\alpha\beta\sigma\pi} = \bar{E}^{\alpha\beta33} = 0$.

In the transversely isotropic case, introducing the potential $\psi(x^\lambda)$ by means of the relation

$$\frac{1}{\bar{h}}\bar{G}_3 U^{\omega} = \varepsilon^{\omega\lambda}\psi_{|\lambda} - \frac{Ah'E'}{1-\nu'^2}\overline{V}_3^{(0)|\alpha\omega} - \frac{A}{\bar{h}}\bar{G}_3\overline{V}_3^{(0)|\omega} +$$

$$+\delta_1\frac{\bar{h}}{2}\left(\frac{\bar{h}}{3}+\frac{h'}{2}\right)\frac{\bar{E}\nu_3}{2(1+\bar{\nu}_3)(1-\bar{\nu})}\overline{V}_3^{(0)}{}_{|\rho}{}^{|\rho\omega} - \frac{h'E'}{2\bar{G}_3(1-\nu'^2)}p_{(0)}{}^{|\omega}, \qquad (29)$$

we obtain the equations

$$\left[\frac{2}{3}\frac{E'\left((\bar{h}+h')^3-\bar{h}^3\right)}{1-\nu'^2} - \delta_1\left(\frac{2\bar{h}^3}{3}+\bar{h}^2h'\right)\frac{\bar{E}\nu_3}{2(1+\bar{\nu}_3)(1-\bar{\nu})}\right]\overline{V}_3^{(0)|\rho\omega}{}_{|\rho\omega} =$$

$$= p_{(0)} - \frac{E'\bar{h}h'}{\bar{G}_3(1-\nu'^2)}p_{(0)}{}^{|\omega}_{|\omega} \qquad \left(\psi - \frac{\bar{h}h'G'}{G_3}\psi^{|\lambda}_{|\lambda}\right)^{|\sigma}_{|\sigma} = 0\Bigg). \qquad (30)$$

For $\delta_1 = 0$, equations (30) are obtained from (21), taking in the latter $\bar{E}/\bar{E} = 0$.

As for the natural boundary conditions, they can readily be obtained in the case of three-layer plates with either a rigid or a light core, using the general formulation given for them in (4).

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