SOME RESULTS IN THE THEORY OF INFINITELY LONG EXPRESSIONS

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Abstract

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SOME RESULTS IN THE THEORY OF INFINITELY LONG EXPRESSIONS

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The paper considers properties of classes defined by systems of axioms of the language $L_{\alpha\beta}$, and solves Scott' s problem.

Let α, β be infinite regular cardinals, $\alpha \geq \beta$. By the first-order language $L_{\alpha\beta}$ we mean (see ([1,8])) the extension of the narrow predicate calculus obtained by allowing conjunctions and disjunctions of fewer than α formulas and quantifications with variables whose number is less than β.

1. A class of models K of a signature σ will be called (α, β)-F-**compact** (see ([2])), where F is some set of axioms of the language $L_{\alpha\beta}$, if for every system A of axioms from F, satisfiability in K of every subsystem A_1 of the system A, $A_1 < \alpha$, entails satisfiability of A in K. In the case when F is the set of all axioms of the language $L_{\alpha\beta}$, we shall call (α, β)-F-compactness simply (α, β)-compactness. We shall call the language $L_{\alpha\beta}$ F-**compact** (**compact**) if the class of all models of the signature of the language $L_{\alpha\beta}$ is (α, β)-F-compact ((α, β)-compact).

The fundamental compactness theorem of Mal' cev in model theory shows the compactness of the language $L_{\omega\omega}$. At the same time, results of Hanf, Tarski ([4]), Karp ([8]), and others show that compactness does not hold for many languages $L_{\alpha\beta}$. In connection with these results it is of interest to find out whether classes axiomatizable in languages $L_{\alpha\beta}$ will be (α, β)-F-compact for certain important sets F. The cases considered are those in which F is the set of universal axioms or the set of positive axioms.

Theorem 1. *Let γ be the cardinality of the signature of the language $L_{\alpha\beta}$, and let 2^γ be less than α. Then the language $L_{\alpha\beta}$ is universally compact.*

Let us note that the class of models of every infinitely long axiom is a projection of the class of models of some universal axiom. In addition, every positive axiom is satisfied on some one-element model. Hence we obtain:

Theorem 2. *Every class of models defined by a single axiom of the language $L_{\alpha\beta}$ is (α, β)-universally (and positively) compact.*

We also note that Theorem 1 does not carry over to the class of models of an arbitrary signature.

Theorems 1 and 2 are used for the proof of Theorems 3 and 4.

2. Will an axiom of the logic $L_{\omega_1\omega}$, whose class of models is closed under taking submodels, be equivalent to a universal axiom (or to a system of such axioms) of the logic $L_{\omega_1\omega}$? This problem was posed by D. Scott in ([3]). A positive answer to this question is given by the following theorem:

An axiom ψ of the logic $L_{\omega_1\omega}$ is preserved under passage to submodels if and only if ψ is equivalent to a universal axiom of the logic $L_{\omega_1\omega}$.

In ([4]) it was shown that in the logic $L_{\alpha\alpha}$, where α is nonmeasurable, there exists an axiom φ such that the class of models of φ is closed under taking submodels, but φ is not equivalent to any system of universal

* When this result had been obtained, the author learned of a brief communication by M. Malitz ([7]), who wrote that he had solved D. Scott' s problem, but did not give the method of solution.

axiom. The following theorem shows that in the case of a strictly inaccessible θ there is no such axiom.

Theorem 3. *An axiom ψ of the logic $L_{\theta\theta}$ (where θ is strictly inaccessible), defining a class K, is equivalent to a universal axiom of the logic $L_{\theta\theta}$ if and only if $S(K) = K$.*

We give a sketch of the proof of Theorem 3. We note that Theorem 4 is proved analogously.

Let M be a set of such universal axioms that $\mathrm{Mod}(A) = \mathrm{Mod}(\psi)$ (see ([3])). Suppose that for no $A \in M$ does $\mathrm{Mod}(A) \subseteq \mathrm{Mod}(\psi)$ hold. Then, by Theorem 2, $\mathfrak{M} \in \mathrm{Mod}(\sim \psi \cup M)$ for some model $\mathfrak{M}$. According to the Löwenheim–Skolem theorem for $L_{\theta\theta}$, the model $\mathfrak{M}$ has a submodel $\mathfrak{M}_1$ of cardinality less than θ such that $\mathfrak{M}_1 \in \mathrm{Mod}(\sim \psi)$. Using the method of diagrams, we establish that there exists an existential axiom λ (of the logic $L_{\theta\theta}$), true in $\mathfrak{M}$, for which $\mathrm{Mod}(\sim \lambda) \supseteq \mathrm{Mod}(\psi)$, which is impossible.

The following result also holds:

An axiom ψ of the logic $L_{\omega_1\omega}$ ($L_{\theta\theta}$) is preserved under passage to extensions if and only if ψ is equivalent to an existential axiom of the logic $L_{\omega_1\omega}$ ($L_{\theta\theta}$).

In the work of Lopez–Escobar ([1]) it was shown that: ($*$) *an axiom of the logic $L_{\omega_1\omega}$ is preserved under passage to homomorphic images if and only if it is equivalent to a positive axiom of the logic $L_{\omega_1\omega}$.*

The result ($*$) was proved by means of the interpolation theorem of the logic $L_{\omega_1\omega}$. The interpolation theorem in the logic $L_{\theta\theta}$, where θ is strictly inaccessible, does not hold (see ([5])). Nevertheless, the following holds.

Theorem 4. *An axiom ψ of the logic $L_{\omega_1\omega}$ ($L_{\theta\theta}$) is equivalent to an existential and positive axiom of the logic $L_{\omega_1\omega}$ ($L_{\theta\theta}$) if and only if ψ holds in all homomorphic images of extensions of models of ψ.*

3. We shall call a class of models K (α, β)-F-**axiomatizable**, where F is some set of axioms of $L_{\alpha\beta}$, if K consists of those and only those models in which a certain system of axioms from F holds. We shall call the language $L_{\alpha\beta}$ **local** if, for every (α, β)-axiomatizable class K, for every model $\mathfrak{N}$, embeddability of each of its submodels $\mathfrak{N}_1$ of cardinality less than α in K entails embeddability of $\mathfrak{N}$ in K.

The well-known Henkin theorem shows the locality of the language $L_{\omega\omega}$. At the same time, Tarski' s results (4) show that the languages $L_{\alpha\alpha}$, where α is singular, are not local. It is unknown whether the languages $L_{\theta\theta}$, where θ is strictly inaccessible, are local. One may hope that characteristics of (θ, θ)-F-axiomatizable classes, even in terms of extensions of mappings, will help solve this problem.

Below we give characteristics of (α, β)-universally axiomatizable classes (where either $\alpha = \beta$, or $\mu < \alpha, \eta < \beta \Rightarrow \mu^\eta < \alpha$), and characteristics of (θ, θ)-axiomatizable classes, where θ is strictly inaccessible*.

Let α, α_i be cardinals ($\alpha \geq \omega$), $\alpha > \alpha_i$, $i = 1, \dots$; $\mathfrak{N}, \mathfrak{M}$ models, $\mathfrak{N}_1 \subseteq S_{\alpha_1}(\mathfrak{N})$. Then: 1) an isomorphic embedding of $\mathfrak{N}_1$ into $\mathfrak{M}$ will be called an $\alpha\alpha(\alpha_1)$-embedding of $\mathfrak{N}$ into $\mathfrak{M}$ relative to $\mathfrak{N}_1$: $\mathfrak{N} \leq_{\alpha\alpha} (\mathfrak{N}_1; \alpha_1)\mathfrak{M}$; 2) an isomorphism φ of the model $\mathfrak{N}_1$ into $\mathfrak{M}$ will be called an $\alpha\alpha(\alpha_1, \alpha_2, \dots, \alpha_{l+1})$-embedding of $\mathfrak{N}$ into $\mathfrak{M}$ relative to $\mathfrak{N}_1$: $\mathfrak{N} \leq_{\alpha\alpha} (\mathfrak{N}_1; \alpha_1, \dots, \alpha_{l+1})\mathfrak{M}$, if for every α_2-extension $\mathfrak{M}_2$ of the model $\varphi(\mathfrak{N}_1) = \mathfrak{M}_1$ in $\mathfrak{M}$ there is an α_2-extension $\mathfrak{N}_2$ of the model $\mathfrak{N}_1$ in $\mathfrak{N}$ and there exists an $\alpha\alpha(\alpha_1 + \alpha_2, \dots, \alpha_{l+1})$-embedding of $\mathfrak{M}_2$ into $\mathfrak{N}_2$ coinciding with φ^{-1} on $\mathfrak{M}_1$.

We shall write $\mathfrak{N} <^\beta_\alpha (\mathfrak{N}_1; \alpha_1)\mathfrak{M}$, $\mathfrak{N}_1 \in S_{\alpha_1}(\mathfrak{N})$, if $\mathfrak{N}_1 <^\beta_\alpha \mathfrak{M}$ in the sense of (1).

The following theorem generalizes the results of (1,4).

Theorem 5. *Let K, K_1 be classes of arbitrary signature σ, $K \subseteq K_1$. Then condition 1) is equivalent to condition 2), and condition 3) to condition 4).*

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* Axioms transferred into $L_{\theta\theta}$ are considered.

1) K is (α, β)-universally axiomatizable in the class K_1, and for all $\mu < \alpha$ and $\eta < \beta$, $\mu^\eta < \alpha$.

2) If, for a model $\mathfrak{A} \in K_1$, for every system $R \subseteq \sigma$, $\overline{\overline{R}} < \alpha$, and every $\mathfrak{A}_1 \in S_{\alpha_1}(\mathfrak{A})$, $\overline{\overline{\mathfrak{A}_1}} = \alpha_1 < \alpha$, there exists such an $\mathfrak{M}^R$ in K_R (see (2)) that $\mathfrak{A}_R \prec_{\alpha\beta} (\mathfrak{A}_{1R}; \alpha_1)\mathfrak{M}^R$, then $\mathfrak{A} \in K$ (and for all $\mu < \alpha$, $\eta < \beta$, $\mu^\eta < \alpha$).

3) K is (α, α)-universally axiomatizable in K_1.

4) If, for $\mathfrak{A} \in K_1$, for every system $R \subseteq \sigma$, $\overline{\overline{R}} < \alpha$, and every $\mathfrak{A}_1 \in S_{\alpha_1}(\mathfrak{A})$, $\overline{\overline{\mathfrak{A}_1}} = \alpha_1 < \alpha$, there exists $\mathfrak{M}^R$ in K_R such that $\mathfrak{A}_R \leq_{\alpha\alpha} (\mathfrak{A}_{1R}; \alpha_1)\mathfrak{M}^R$, then $\mathfrak{A} \in K$.

The proofs of the theorems are carried out by the method of ([5]).

Theorem 6. Let K, K_1 be classes of models of signature σ, $\overline{\overline{\sigma}} < \theta$, $K \subseteq K_1$. Then the following conditions are equivalent:

1) K is (θ, θ)-$\underbrace{\forall\exists\ldots}_{l}$-axiomatizable in K_1, where θ is strongly inaccessible.

2) If, for a model $\mathfrak{A} \in K_1$, any tuple $(a_1, \ldots, a_l)$ of length l, $a_i < \theta$, $i = 1, \ldots, l$, and any $\mathfrak{A}_1 \in S_{\alpha_1}(\mathfrak{A})$ in K, there is an $\mathfrak{M}$ such that $\mathfrak{A} \leq_{\theta\theta} (\mathfrak{A}_1; \alpha_1, \ldots, \alpha_l)\mathfrak{M}$, then $\mathfrak{A} \in K$.

It is not difficult to show that the class of models $\mathfrak{M}$ satisfying the condition $\mathfrak{A} \leq_{\theta\theta} (\mathfrak{A}_1; \alpha_1, \ldots, \alpha_l)\mathfrak{M}$ is described by an axiom of the form $\exists\forall\ldots$ of the logic $L_{\theta\theta}$.

Hence it follows that the model $\mathfrak{A}$ will be an $F^{\theta\theta}_{A_l}$-touching point of the class K (see ([6])) if and only if $\mathfrak{A}$ satisfies condition 2), which proves theorem 6.

Let us note that a model will be an $F^{\theta\theta}_{A}$-touching point of the class K if and only if $\mathfrak{A}$ is an $F^{\theta\theta}_{A_l}$-touching point (or an $F^{\theta\theta}_{E_l}$-touching point) for any $l \in \omega$. Hence we obtain:

Theorem 7. Let K be a class of signature σ, $\overline{\overline{\sigma}} < \theta$, where θ is strongly inaccessible. Then the following conditions are equivalent:

1) K is (θ, θ)-axiomatizable.

2) If, for a model $\mathfrak{A}$, any tuple $(\alpha_1, \ldots, \alpha_l)$, and any $\mathfrak{A}_1 \in S_{\alpha_1}(\mathfrak{A})$ in K, there is an $\mathfrak{M}$ such that $\mathfrak{A} \leq_{\theta\theta} (\mathfrak{A}_1; \alpha_1, \ldots, \alpha_l)\mathfrak{M}$, then $\mathfrak{A} \in K$.

3) If, for a model $\mathfrak{A}$, any tuple $(\alpha_1, \ldots, \alpha_l)$, there is in K an $\mathfrak{M}$ such that $\mathfrak{M} \leq_{\theta\theta} (\alpha_1, \ldots, \alpha_l)\mathfrak{A}$, then $\mathfrak{A} \in K$.

In addition, using the results of ([6]), one can obtain a large series of characteristics of (θ, θ)-axiomatizable classes.

Proposition 1. Let $\mathfrak{A}, \mathfrak{M}$ be models. Then:

1) $\mathfrak{A}$ and $\mathfrak{M}$ are $L_{\alpha\alpha}$-universally equivalent if and only if
$\mathfrak{A} \leq_{\alpha\alpha} [1]\mathfrak{M}$ and $\mathfrak{M} \leq_{\alpha\alpha} [1]\mathfrak{A}$.

2) $\mathfrak{A}$ and $\mathfrak{M}$ are $L_{\theta\theta}$-elementarily equivalent if and only if
$\mathfrak{A} \leq_{\theta\theta} [l]\mathfrak{M}$ and $\mathfrak{M} \leq_{\theta\theta} [l]\mathfrak{A}$ for any $l \in \omega$ (where θ is strongly inaccessible).

Here $\mathfrak{A} \leq_{\alpha\alpha} [l]\mathfrak{M}$ denotes that for any sequence $(\alpha_1, \ldots, \alpha_l)$ and any $\mathfrak{A}_1 \in S_{\alpha_1}(\mathfrak{A})$,
$\mathfrak{A} \leq_{\alpha\alpha} (\mathfrak{A}_1; \alpha_1, \ldots, \alpha_l)\mathfrak{M}$.

From proposition 1 it follows that

$L_{\alpha\alpha}$-universal and $L_{\theta\theta}$-elementary equivalences are preserved under direct products (cf. ([1])).

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Note: Figure translations are in progress. See original paper for figures.