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Abstract

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A. I. GUSEINOV, G. A. RASULOVA

STUDY OF A MIXED PROBLEM FOR ONE CLASS OF QUASILINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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In the present work we study the mixed problem A

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} - a\frac{\partial^3 U}{\partial t\,\partial x^2} = F[t, x, U, U_x, U_t, U_{xx}, U_{tx}], \tag{1}$$

$$U(t, 0) = U(t, \pi) = 0, \qquad U(0, x) = \varphi(x), \qquad U_t(0, x) = \psi(x), \tag{2}$$

where $0 \le x \le \pi$; $0 \le t \le T < \infty$; $a > 0$ is a fixed number; F, φ, ψ are prescribed functions. By means of the method of successive approximations and the strengthened Schauder principle (see ([1]), p. 206), nonlocal theorems on existence and uniqueness of a generalized, almost everywhere, and classical solution of problem A are proved; the continuous dependence of all three types of solutions of problem A on the initial data and on the right-hand side of equation (1) is studied; in addition, the boundedness (in a certain metric) and the behavior as $t \to +\infty$ of solutions of problem A are studied when these solutions exist for every $T > 0$.

We adopt the following notation and definitions. Denote by

$$B_T^{\alpha_0,\ldots,\alpha_l}$$

the set of all functions of the form

$$U(t, x) = \sum_{n=1}^{\infty} U_n(t) \sin nx,$$

considered in the domain $D_T(0 \leq t \leq T,\ 0 \leq x \leq \pi)$, where each of the functions $U_n(t)$ is $l \geq 0$ times continuously differentiable on the interval $[0, T]$ and

$$\sum_{n=1}^{\infty} \left\{ n^{\alpha_i} \max_{0 \leq t \leq T} \left| U_n^{(i)}(t) \right| \right\}^2 < \infty \qquad (\alpha_i \geq 1,\ i = 0, \dots, l).$$

In this set we define the norm as follows:

$$\|U(t,x)\|_{B_T^{\alpha_0,\dots,\alpha_l}} = \sum_{i=0}^{l} \left\{ \sum_{n=1}^{\infty} \left[n^{\alpha_i} \max_{0 \leq t \leq T} \left| U_n^{(i)}(t) \right| \right]^2 \right\}^{1/2}. \tag{3}$$

It is obvious that this space is a Banach space.

Denote by E_i $(i \geq 1)$ the set of all functions $f(x)$ that are $i-1$ times continuously differentiable on $[0, \pi]$, with $f^{(i)}(x) \in L_2(0, \pi)$, and $f^{(2s)}(0) = f^{(2s)}(\pi) = 0$, where $s = 0, \dots, [(i-1)/2]$.

Definition 1. A **generalized solution of problem A** is a function $U(t, x)$ that belongs to $B_T^{2,1}$, assumes all values (2) in the ordinary sense, and satisfies the integral identity

$$\int_0^T \int_0^\pi \left\{ \frac{\partial U}{\partial t} \frac{\partial V}{\partial t} + \frac{\partial^2 U}{\partial x^2} V - a \frac{\partial^2 U}{\partial t\, \partial x} \frac{\partial V}{\partial x} + F[t, x, U, U_x, U_t, U_{xx}, U_{tx}] V \right\} dx\, dt +$$

$$+ \int_0^\pi \psi(x) V(0, x)\, dx = 0$$

for every function $V(t, x)$ continuously differentiable in the domain D_T and satisfying the conditions

$$V(t, 0) = V(t, \pi) = V(T, x) = 0.$$

Definition 2. By an **almost-everywhere solution of problem A** we shall mean a function $U(t, x)$, continuous in the closed domain D_T together with all its derivatives entering equation (1), with the possible exception of the derivatives U_{tt} and U_{tx^2}, which belong to $L_2(D_T)$, assume all the values (2) in the usual sense, and satisfy equation (1) almost everywhere in D_T.

Definition 3. By a **classical solution of problem A** we shall mean a function $U(t, x)$ which is continuous in the closed domain D_T together with all its derivatives entering equation (1), and satisfies all conditions (1) and (2) in the usual sense.

We give some of the results obtained.

Generalized solution of problem A

Theorem 1. Suppose: 1) $\varphi(x) \in E_2$, $\psi(x) \in E_1$; 2) the function $F[t,x,U_1,\dots,U_5]$, defined in the domain

$$D_T \times (|U_i| < \infty) \qquad (i = 1,\dots,5),$$

is continuous with respect to the totality of its variables and

$$|F[t,x,U_1,\dots,U_5] - F[t,x,\widetilde{U}_1,\dots,\widetilde{U}_5]| \le$$

$$\le a(t,x)\sum_{k=1}^{3}|U_k - \widetilde{U}_k| + b(t)\sum_{k=4}^{5}|U_k - \widetilde{U}_k|,$$

where $a(t,x) \in L_2(D_T)$ and $b(t) \in L(0,T)$.

Then problem A has a unique generalized solution $U(t,x)$, which can be found by the method of successive approximations; moreover, the convergence of the successive approximations $U_k(t,x)$ to $U(t,x)$ is characterized by the following inequality for $i=2,\ j=1$:

$$\|U_k(t,x) - U(t,x)\|_{B_T^{i,j}} \le a_T b_T^k/\sqrt{k!} \qquad (k = 0,1,2,\dots), \qquad (4_{i,j})$$

where a_T, b_T are definite constants depending only on T.

Everywhere below, the problem A corresponding to the data $\widetilde{F},\ \widetilde{\varphi},\ \widetilde{\psi}$ will be called problem $\widetilde{A}$.

Theorem 2. Suppose: 1) all the conditions of Theorem 1 are fulfilled, and the functions $\widetilde{\varphi},\ \widetilde{\psi},\ \widetilde{F}$ satisfy the same (corresponding) conditions; 2) in the domain

$$D_T \times \left(|U_i| \le \frac{\pi}{\sqrt{6}}R\right) \times \left(|U_j| < \infty\right) \qquad (i = 1,2,3; \quad j = 4,5)$$

$$|F[t,x,U_1,\dots,U_5] - \widetilde{F}[t,x,U_1,\dots,U_5]| \le f(t,x),$$

where $f(t,x) \in L_2(D_T)$, and $R > 0$ is a finite number with which all generalized solutions of problem A are a priori estimated in the sense

$$\|U(t,x)\|_{B_T^{2,1}} \le R^*.$$

Then for the unique generalized solutions $U(t,x)$ and $\widetilde{U}(t,x)$, respectively of the problems A and $\widetilde{A}$, one has

$$\|U(t,x) - \widetilde{U}(t,x)\|_{B_T^{2,1}} \le C\left\{\|\varphi'' - \widetilde{\varphi}''\|_{L_2(0,\pi)} + \|\psi' - \widetilde{\psi}'\|_{L_2(0,\pi)} + \|f\|_{L_2(D_T)}\right\},$$

where $C > 0$ is some constant.

Theorem 3. If conditions 1) and 2) (for $T = +\infty$) of Theorem 1 are fulfilled and $F[t,x,0,\dots,0] \in L_2(D_\infty)$, then for the function $U(t,x)$ (where $U(t,x) = U_T(t,x)$

for $(t,x) \in D_T$ and $U_T(t,x)$ is the unique generalized solution of problem A in D_T)

$$\sup_{T>0} \|U(t,x)\|_{B_T^{2,1}} \equiv C_0 < \infty$$

and in inequality $(4_{2,1})$ the index T at the coefficients a_T and b_T may be omitted. If, in addition, it is assumed that

$$\lim_{T\to+\infty} \left\{ Te^{-T} \left[\|e^t F[t,x,0,\dots,0]\|_{L_2(D_T)} + \|e^t a(t,x)\|_{L_2(D_T)} + \|e^t b(t)\|_{L_2(0,T)} \right] \right\} = 0, \quad (5)$$

* The existence of such a finite R is easily established with the aid of Gronwall' s inequality.

where $\varepsilon = \min\{1/\alpha, \alpha/2\}$, then all the functions $\partial^s U/\partial t^i \partial x^j$ $(s = 0,1)$ tend to zero as $t \to +\infty$ uniformly with respect to $x \in [0,\pi]$.

Theorem 4. Suppose: 1) condition 1) of Theorem 1 is fulfilled; 2) $F \equiv F[t,x,U,U_x,U_t]$, and the function $F[t,x,U_1,U_2,U_3]$, defined in the domain $D_T \times (|U_i| < \infty)$ $(i = 1,2,3)$, is continuous in the totality of its variables and

$$|F[t,x,U_1,U_2,U_3]| \le a(t,x) \sum_{i=1}^{3} |U_i| + b(t,x),$$

where $a(t,x), b(t,x) \in L_2(D_T)$.

Then problem A has a generalized solution.

Solution almost everywhere of problem A

Theorem 5. Suppose: 1) $\varphi(x) \in E_3$, $\psi(x) \in E_2$; 2) $F \equiv F[t,x,U,U_x,U_t]$, and the function $F[t,x,U_1,U_2,U_3]$, defined in the domain $D_T \times (|U_i| < \infty)$ $(i = 1,2,3)$, is continuous in the totality of its variables,

$$F[t,0,0,U_2,0] = F[t,\pi,U_2,0] \equiv 0,$$

$$|F[t,x,U_1,U_2,U_3] - F[t,x,\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3]| \le a(t,x) \sum_{i=1}^{3} |U_i - \widetilde{U}_i|,$$

where $a(t,x) \in L_2(D_T)$; 3) in the domain

$$\Omega_R \equiv D_T \times \left(|\xi_i| \le \frac{\pi}{\sqrt{6}} R \right) \quad (i = 2,3,4):$$

a) the functions $\partial F[t,\xi_1,\dots,\xi_4]/\partial \xi_j$ $(j = 1,2,3,4)$ are continuous in the totality of their variables;

b)

$$\left|\frac{\partial F[t,\xi_1,\xi_2,\xi_3,\xi_4]}{\partial \xi_i}-\frac{\partial F[t,\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3,\tilde{\xi}_4]}{\partial \xi_i}\right|\le \{a(t,\xi_1)\operatorname{sign}[(3-i)(4-i)]+b(t)\operatorname{sign}[(1-i)(2-i)]\}\sum_{j=2}^{4}|\xi_j-\tilde{\xi}_j|,$$

where $i=1,2,3,4$, $a(t,x)\in L_2(D_T)$, $b(t)\in L_2(0,T)$, and $R>0$ is a finite number with which (under conditions 1) and 2) of this theorem) all generalized solutions (and, in particular, solutions almost everywhere) of problem A are a priori estimated in the sense

$$\|U(t,x)\|_{B_T^{2,1}}\le R.$$

Then problem A has a unique solution almost everywhere $U(t,x)$, which can be found by the method of successive approximations, and the convergence of the successive approximations $U_k(t,x)$ to $U(t,x)$ is characterized by inequality $(4_3,2)$.

Theorem 6. Suppose: 1) all the conditions of Theorem 5 are fulfilled, and the functions $\widetilde{\varphi},\widetilde{\psi}$ and $\widetilde{F}$ satisfy the same (corresponding) conditions; 2) in the domain Ω_R,

$$|F-\widetilde{F}|\le f(t,x),$$

$$\left|\frac{\partial F}{\partial \xi_i}-\frac{\partial \widetilde{F}}{\partial \xi_i}\right|\le f(t,\xi_1)\quad (i=1,2),$$

$$\left|\frac{\partial F}{\partial \xi_j}-\frac{\partial \widetilde{F}}{\partial \xi_j}\right|\le g(t)\quad (j=3,4),$$

where $f(t,x)\in L_2(D_T)$ and $g(t)\in L_2(0,T)$.

Then for the unique solutions almost everywhere $U(t,x)$, $\widetilde{U}(t,x)$ of problems A and $\widetilde{A}$, respectively, one has

$$\|U(t,x)-\widetilde{U}(t,x)\|_{B_T^{3,2}}\le C_T\{\|\varphi'''-\widetilde{\varphi}'''\|_{L_2(0,\pi)}+\|\psi'-\widetilde{\psi}'\|_{L_2(0,\pi)}+\|f\|_{L_2(D_T)}+\|g\|_{L_2(0,T)}\},$$

where $C_T>0$ is a constant depending only on T.

Theorem 7. Suppose: 1) conditions 1) and 2) (for $T=+\infty$) of Theorem 5 are fulfilled; 2) the functions $\partial F[t,\xi_1,\dots,\xi_4]/\partial\xi_i$ $(i=1,2,3,4)$ are continuous in the totality of their variables in the domain $D_\infty \times (|\xi_j| < \infty)$ $(j = 2, 3, 4)$; 3) for every $C > 0$* in the domain $D_\infty \times (|\xi_i| \le C)$ $(i = 2, 3, 4)$

$$|\partial F / \partial \xi_j| \le f_C(t, \xi_1) \quad (j = 1, 2)$$

and

$$|\partial F / \partial \xi_k| \le g_C(t) \quad (k = 3, 4),$$

where $F[t, x, 0, \dots, 0]$, $f_C(t, x) \in L_2(D_\infty)$ and $g_C(t) \in L_2(0, \infty)$.

Then the function $U(t, x)$ (where $U(t, x) = U_T(t, x)$ for $(t, x) \in D_T$; $U_T(t, x)$ is the unique solution almost everywhere of problem A in D_T) is

* It is sufficient to require fulfillment of this condition only for C_0, where C_0 is the designation of Theorem 3.

in the domain D_∞ the unique almost-everywhere solution of problem A,

$$\sup_{T>0} \|U(t, x)\|_{B_T^{3,2}} < \infty,$$

and in inequality $(4_{3,2})$ the index T on the coefficients a_T and b_T may be omitted. If, in addition, condition (5) is fulfilled (for $b(t) \equiv 0$), then the functions $\partial^s U / \partial x^i \partial t^j$ $(s = 0, 1, 2,\ i = 0, 1, 2,\ j = 0, 1)$ tend to zero as $t \to +\infty$ uniformly with respect to $x \in [0, \pi]$.

Theorem 8. Suppose: 1) condition 1) of Theorem 5 is fulfilled; 2) condition 2) of Theorem 4 is fulfilled and

$$F[t, 0, 0, U_2, 0] = F[t, \pi, 0, U_2, 0] \equiv 0;$$

3) the functions

$$\partial F[t, \xi_1, \dots, \xi_4] / \partial \xi_i \quad (i = 1, \overline{4})$$

are continuous jointly in their variables in the domain $D_T \times (|\xi_j| < \infty)$ $(j = 2, 3, 4)$.

Then problem A has a unique almost-everywhere solution.

Classical solution of problem A

Theorem 9. Suppose: 1) $\varphi(x) \in E_3,\ \psi(x) \in E_3$; 2) condition 2) of Theorem 5 is fulfilled; 3) in the domain

$$D_T \times \left(|\xi_i| \le \frac{\pi}{\sqrt{6}} R \right) \quad (i = 2, 3, 4):$$

a) the functions

$$\partial^{sF}[t, \xi_1, \dots, \xi_4] / \partial \xi_1^{\alpha_1} \cdots \partial \xi_4^{\alpha_4} \quad (s = 1, 2)$$

are continuous jointly in their variables; b)

$$\left|\partial^2 F[t,\xi_1,\tilde{\xi}_2,\tilde{\xi}_3,\tilde{\xi}_4]/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}-\partial^2 F[t,\xi_1,\xi_2,\xi_3,\xi_4]/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}\right|\le\tilde{b}(t,\xi_1)\sum_{i=2}^{4}|\xi_i-\tilde{\xi}_i|,$$

where $\tilde{b}(t,x)\in L_2(D_T)$ and R is the notation of Theorem 5.

Then problem A has a unique classical solution $U(t,x)$, which can be found by the method of successive approximations; moreover the convergence of the successive approximations $U_k(t,x)$ to $U(t,x)$ is characterized by inequality $(4_{3,3})$.

Theorem 10. Suppose: 1) conditions 1), 2) and 3a) of Theorem 9 are fulfilled, and the functions $\tilde{\varphi},\tilde{\psi}$ and $\tilde{F}$ satisfy the same (corresponding) conditions; 2) in the domain Ω_R the condition

$$\left|\partial^2 F/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}-\partial^2\tilde{F}/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}\right|\le f(t,\xi_1)$$

is fulfilled, as well as condition 2) of Theorem 6.

Then for the unique classical solutions $U(t,x)$ and $\tilde{U}(t,x)$ of problems A and $\tilde{\text{A}}$, respectively, we have

$$\|U(t,x)-\tilde{U}(t,x)\|_{B_T^{3,3,1}}\le C_T\{\|\varphi'''-\tilde{\varphi}'''\|_{L_2(0,\pi)}+\|\psi'''-\tilde{\psi}'''\|_{L_2(0,\pi)}+\|f\|_{L_2(D_T)}+\|g\|_{L_2(0,T)}\}.$$

Theorem 11. Suppose: 1) condition 1) of Theorem 9 and condition 2) (for $T=+\infty$) of Theorem 5 are fulfilled; 2) the functions

$$\partial^{sF}[t,\xi_1,\dots,\xi_4]/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}\quad(s=1,2)$$

are continuous jointly in their variables in the domain $D_\infty\times(|\xi_i|<\infty)$ $(i=2,3,4)$; 3) for every $C>0$, in the domain $D_\infty\times(|\xi_i|\le C)$ $(i=2,3,4)$,

$$|\partial^2 F/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}|\le f_C(t,\xi_1),$$

and condition 3) of Theorem 7 is fulfilled.

Then the function $U(t,x)$ (where $U(t,x)=U_T(t,x)$ for $(t,x)\in D_T$, and $U_T(t,x)$ is the unique classical solution of problem A in D_T) is, in the domain D_∞, the unique classical solution of problem A,

$$\sup_{T>0}\|U(t,x)\|_{B_T^{3,3}}<\infty,$$

and in inequality $(4_{3,3})$ the index T on the coefficients a_T and b_T may be omitted. If, in addition, condition (5) is fulfilled (for $b(t)\equiv 0$), then the functions $\partial^{sU}/\partial x^i\partial t^j$ $(s=0,1,2,3,\ i=0,1,2,\ j=0,1)$ tend to zero as $t\to+\infty$ uniformly with respect to $x\in[0,\pi]$.

Theorem 12. Suppose: 1) condition 1) of Theorem 9 and condition 2) of Theorem 8 are fulfilled; 2) the functions

$$\partial^{sF}[t,\xi_1,\dots,\xi_4]/\partial\xi_1^{\alpha_1}\cdots\partial\xi_4^{\alpha_4}\quad(s=1,2)$$

are continuous jointly in their variables in the domain $D_T \times (|\xi_i| < \infty)$ $(i = 2, 3, 4)$.

Then problem A has a unique classical solution.

Institute of Cybernetics
Academy of Sciences of the Azerbaijan SSR

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Note: Figure translations are in progress. See original paper for figures.