ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A GROWING POTENTIAL IN DOMAINS WITH INFINITE BOUNDARY

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Abstract

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ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A GROWING POTENTIAL IN DOMAINS WITH INFINITE BOUNDARY

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Consider the problem

$$\{-\Delta + q(x) - \lambda\}u = 0 \quad \text{in } D \subset E_3; \tag{1}$$

$$u|_\Gamma = 0. \tag{2}$$

It is assumed that $q(x) \geq 1$ is a real continuous function,

$$\lim_{|x|\to\infty} q(x) = +\infty. \tag{3}$$

In the present paper we study the asymptotics of the number $N(\lambda)$ of eigenvalues of the operator of problem (1)–(2) in an infinite domain D with infinite boundary Γ, not exceeding λ. We single out a class Ω of domains D for which the problem posed can be solved.

We shall say that the domain $D \subset \Omega$ if conditions I–III are satisfied.

I. The surface Γ is Lyapunov. The Lyapunov constants are uniformly bounded when the center of the Lyapunov sphere runs over the surface Γ.

II.

$$\sup_{s_0\subset\Gamma} \text{mes}\{s: \ s\subset\Gamma, \ \delta \leq R \leq |s-s_0| \leq R+1\} < Ce^{aR},$$

where $R > 0$ is an arbitrary number, $\delta > 0$ is the radius of the Lyapunov sphere for the surface Γ, and $a > 0, \ C > 0$ do not depend on R.

III.

$$\text{mes}\{x:\ x \subset D,\ |x| = r,\ \rho(x) \le \varepsilon\} \le C\varepsilon\ \text{mes}\{x:\ x \subset D,\ |x| = r,\ \rho(x) \ge \varepsilon\}$$

for almost all r, where $C > 0$ does not depend on r and ε, $0 < \varepsilon \le \varepsilon_0$, $\varepsilon_0 > 0$, and $\rho(x)$ is the distance from the point x to the boundary Γ.

To formulate the first result, consider the problem

$$\{-\Delta + q_a(x) - \lambda\}u = 0 \quad \text{in } E_3, \tag{4}$$

where

$$q_a(x) = \begin{cases} q(x), & x \subset D, \\ a + \max\limits_{|x|=r,\ x\subset D} |q(x)|, & x \subset E_3/D, \quad a = \text{const} > 0. \end{cases} \tag{5}$$

If the domain D satisfies condition I and the potential $q(x)$ satisfies condition (3), then the spectra of problems (1)–(2) and (4) are discrete. Denote by λ_n, $\psi_n(x)$, $\lambda_{n,a}$, $\psi_{n,a}(x)$ the eigenvalues and eigenfunctions of problems (1)–(2) and (4), respectively.

Theorem 1. *Under the assumptions made, all eigenvalues and eigenfunctions of problem (1)–(2) can be obtained as limit points of the sequences $\lambda_{n,a}$ and $\psi_{n,a}(x)$ as $a \to +\infty$.*

The heuristic method for finding the leading term of the asymptotics of $N(\lambda)$ is as follows. Since for the function $N_a(\lambda)$—the number of eigenvalues—

eigenvalues of problem (4)—the formula holds

$$N_a(\lambda) \sim \frac{1}{6\pi^2} \int_{\{x: x\subset E_3,\ q_a(x)<\lambda\}} \{\lambda - q_a(x)\}^{3/2}\, dx, \qquad \lambda \to +\infty, \tag{6}$$

then for $N(\lambda)$ the formula

$$N(\lambda) \sim \frac{1}{6\pi^2} \int_{\{x: x\subset D,\ q(x)<\lambda\}} \{\lambda - q(x)\}^{3/2}\, dx, \qquad \lambda \to +\infty. \tag{7}$$

must be true.

We were unable to justify the passage, as $a \to +\infty$, from formula (6) to formula (7). We shall prove formula (7) by the method of T. Carleman–E. Titchmarsh–B. M. Levitan ([1]). For this we shall need the following result, of independent interest.

Lemma 1. Let $D \subseteq \Omega$, suppose condition (3) is fulfilled and the inequality

$$\max_{|x|=r,\ y\subset D} q(x) \le C \min_{|x|=r,\ x\subset D} q(x), \tag{8}$$

where $C \ge 1$ does not depend on r.

Denote by $G(x, y, \mu)$ the resolvent kernel of the Laplace operator for the Dirichlet problem in the domain D:

$$(-\Delta + \mu^2)G(x, y, \mu) = \delta(x - y) \quad \text{in } D, \qquad G|_\Gamma = 0. \tag{9}$$

Then

$$0 \le G(x, y, \mu) \le \frac{\exp(-\mu|x - y|)}{4\pi|x - y|}, \qquad x, y \subset \overline{D}, \tag{10}$$

$$G(x, y, \mu) = \frac{\exp(-\mu|x - y|)}{4\pi|x - y|}(1 + r(y, \mu)), \qquad \mu \to +\infty, \tag{11}$$

where the estimate is uniform with respect to x and y varying in the domain D, $|x - y| \le \frac{1}{2}\max[\rho(y), \rho(x)]$,

$$r(y, \mu) = O\left(\frac{\exp(-\mu\rho(y))}{\rho^2(y)}\right). \tag{12}$$

Put

$$\chi = \sqrt{\mu^2 + q(y)}. \tag{13}$$

Then

$$\int_D G^2(x, y, \chi)\, dx = \frac{1}{8\pi\chi}\,(1 + O(r(y, \chi))). \tag{14}$$

Moreover,

$$\iint_{DD} G^2(x, y, \chi)\, dx\, dy = \int_D \frac{dy}{8\pi\chi}\left(1 + O\left(\frac{1}{\mu^\gamma}\right)\right), \qquad \mu \to +\infty,\ \gamma > 0. \tag{15}$$

For any natural number $p > 0$ the estimate

$$\frac{\partial^p G(x,y,\mu)}{(\partial\mu)^{2p}} = \frac{\partial^p}{(\partial\mu)^{2p}}\frac{\exp(\mu|x-y|)}{4\pi|x-y|}(1+r_1(y,\mu)), \qquad \mu\to+\infty, \tag{16}$$

holds, where

$$r(y,\mu) = O\left(\frac{\mu^{2p-1}\exp(-\delta\mu\rho(y))}{\rho^{p-1}(y)}\right), \qquad \delta>0, \quad |x-y|\le\frac{1}{2}\max[\rho(x),\rho(y)].$$

In proving Lemma 1, the methods of the papers ([3,4]) are used. We introduce assumptions on the potential which coincide with the assumptions made in ([1]) for the case $D=E_3$:

$$q(x) < C\exp(c|x|), \qquad c>0, \quad C>0; \tag{17}$$

$$|q(x)-q(y)| < Cq^a(x)|x-y|, \qquad |x-y|\le 1, \quad 0<a<3/2; \tag{18}$$

$$q(y) < C\exp\left(\frac{1}{2}|x-y|\sqrt{q(x)}\right), \quad |x-y|>1; \tag{19}$$

$$\int_D \frac{dx}{q^A(x)} < \infty, \quad A>0. \tag{20}$$

With the aid of Lemma 1, using the methods set forth in the book ([1]), one can obtain the relation

$$\int_0^\infty \frac{dN(t)}{(t+\mu)^{2p}} = \int_0^\infty \frac{d\varphi(t)}{(t+\mu)^{2p}}\left(1+O\left(\left(\frac{1}{\mu^\gamma}\right)\right)\right), \quad \mu\to+\infty, \tag{21}$$

where p is an arbitrary natural number,

$$\varphi(t) = \frac{1}{6\pi^2}\int_0^\infty (t-\tau)^{3/2}\,d\zeta(\tau), \tag{22}$$

$$\sigma(t) = \operatorname{mes}\{x:\ x\subset D,\ q(x)<t\}.$$

If we assume that for sufficiently large t the inequality

$$\sigma(t) < C\sigma(t/2), \tag{23}$$

is satisfied, then, as shown in ([1]), p. 525, the Tauberian condition sufficient for applying to equality (20) the Tauberian theorem with a remainder term from ([2]) will be satisfied. As a result we obtain the theorem:

Theorem 2. Let $D \subset \Omega$, and let conditions (3), (8), (17)–(20), (23) be satisfied. Then

$$N(\lambda) = \frac{1}{6\pi^2} \int_{\{x\colon\ x \subset D,\ q(x) < \lambda\}} \{\lambda - q(x)\}^{3/2}\, dx \left(1 + O\left(\frac{\lambda}{\ln \lambda}\right)\right), \quad \lambda \to +\infty. \tag{24}$$

We note that condition (23) will be satisfied if the domain D contains a cone K_0 of arbitrarily small but fixed aperture and if, for sufficiently large $|x| = r$, the inequality

$$a_0 r^k < q(x) < a_1 r^k, \tag{25}$$

holds, where k, a_0, a_1 are positive constants. If estimate (25) holds and the boundary Γ is contained between the paraboloids

$$z = a_2(x^2 + y^2)^p, \quad z = a_3(x^2 + y^2)^p, \quad 0 < a_2 < a_3, \quad p > 1/2, \tag{26}$$

then estimate (23) will also be satisfied.

Using the method of B. M. Levitan (([1]), p. 511), one can show that under the conditions of Theorem 2 the formula

$$N_\tau(\lambda) = \frac{1}{6\pi^2} \int_{\{x\colon\ x \subset D,\ q(x) < \lambda\}} q^\tau(x) \{\lambda - q(x)\}^{3/2}\, dx \left(1 + O\left(\frac{1}{\ln \lambda}\right)\right), \quad \lambda \to +\infty; \tag{27}$$

is valid, where

$$N_\tau(x) = \sum_{\lambda_n < \lambda} a_n^{(\tau)}, \quad a_n^{(\tau)} = \int_D q^\tau(x) \psi_n^2(x)\, dx. \tag{28}$$

Consider a solution of equation (1) satisfying the boundary condition

$$\partial u / \partial n + \sigma u|_\Gamma = 0, \tag{29}$$

where $\sigma(s) \subset C_1(\Gamma)$, and n is the exterior normal. Denote by $N_\sigma(\lambda)$ and $M(\lambda)$ the number of eigenvalues of problem (1), (29) for $\sigma \geq 0$ and $\sigma \equiv 0$, respectively. From the variational principle it follows that

$$N(\lambda) \leq N_\sigma(\lambda) \leq M(\lambda). \tag{30}$$

Therefore, in order to prove the asymptotic formula (24) for the function $N_\sigma(\lambda)$, it is sufficient to prove formula (24) for the function $M(\lambda)$.

Remark 1. If $q(x) \geqslant 1$ is a continuous function, and the boundary of the domain D satisfies conditions I, II, then the symmetric operator defined by the differential expression (1) on the set of twice differentiable functions finite near infinity and satisfying condition (2), as can be shown, is essentially self-adjoint (has zero deficiency indices). In order to prove the analogous assertion for the symmetric operator corresponding to the boundary condition (29), we had to make an additional assumption, consisting of the following:

IV. Outside a sphere of arbitrarily large, but fixed, radius the boundary of the domain D is star-shaped with respect to the point O introduced in condition I.

In a somewhat different in form, but essentially equivalent, formulation this condition was introduced in another problem in the papers ([4]). The results formulated below are valid for any self-adjoint extension of the symmetric operator defined on the set of functions twice differentiable, finite near infinity, and satisfying condition (29).

Theorem 3. *Under the conditions of Theorem 2, for the function $M(\lambda)$, and consequently, by virtue of inequality (30), also for the function $N_\sigma(\lambda)$, formula (24) is valid.*

The proof of Theorem 3 is based on the lemma:

Lemma 2. *Under the conditions of Lemma 1, the solution of the problem*

$$(-\Delta + \mu^2)\mathcal{G}(x, y, \mu) = \delta(x - y) \text{ in } D, \qquad \frac{\partial \mathcal{G}}{\partial n} + \sigma\mathcal{G}\big|_\Gamma = 0; \tag{31}$$

satisfies the estimates (11), (14), (15), (16) and the estimate

$$|\mathcal{G}(x, y, \mu)| < C\exp(-\beta\mu|x - y|)/|x - y|, \qquad x, y \subset \overline{D}, \tag{32}$$

where $0 < \beta < 1$ may be taken arbitrarily close to 1.

Remark 1. It is possible that estimate (32) is valid with $\beta = 1$, but we have not been able to prove this. For the proof of Theorem 3, the assertion of Lemma 2 is quite sufficient.

Remark 2. Formula (24) (without the remainder term), for the operator of the Dirichlet problem, apparently can be obtained under other assumptions by the method of paper ([5]), as B. M. Levitan and A. A. Arsen'ev indicated. However, by the method of paper ([5]) it is not possible to consider other boundary conditions.

In conclusion we note that the proof is essentially based on the fact that, under the conditions of Lemma 1, the integral equation of potential theory

$$\frac{1}{2}b(s) - \int_\Gamma \frac{\partial}{\partial n_s}\frac{\exp(-\mu|s-t|)}{4\pi|s-t|}\, b(t)\, dt = b_0(s) \tag{33}$$

as $\mu \to +\infty$ is an equation with a "small kernel"; more precisely, the norm of the integral operator in $C(\Gamma)$ is $O(\mu^{-\alpha})$, where $\alpha > 0$ is the Lyapunov exponent of the surface Γ.

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Note: Figure translations are in progress. See original paper for figures.