

ON THE STABILITY OF SMALL PERIODIC SOLUTIONS AT MULTIDIMENSIONAL DEGENERACIES

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Abstract

Full Text

MATHEMATICS

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ON THE STABILITY OF SMALL PERIODIC SOLUTIONS AT MULTIDIMENSIONAL DEGENERACIES

(Presented by Academician A. Yu. Ishlinskii on 20 XI 1967)

1. Consider the system of ordinary differential equations

$$dx/dt = f(t, x, \lambda) \tag{1}$$

with right-hand sides that are ω -periodic in t and depend on the scalar parameter λ . We shall assume that the right-hand sides possess sufficient smoothness with respect to the aggregate of the variables.

Throughout the paper it is assumed that system (1), for all λ , has the zero solution: $f(t, 0, \lambda) \equiv 0$. We shall be interested in the question of the existence and stability of small nonzero ω -periodic solutions of system (1) (see ^(1,2)).

We shall call a number λ_0 (see ^(3,4)) a bifurcation value of the parameter λ if to every $\varepsilon > 0$ there corresponds a number $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for which system (1) has a nonzero ω -periodic solution whose amplitude $\max \|x(t)\|$ does not exceed ε .

Denote by $V(t, \lambda)$ the fundamental matrix (see ⁽⁵⁾) of the system linearized at zero,

$$dx/dt = A(t, \lambda)x. \tag{2}$$

It is easy to see (see ^(3,5)) that only those values of the parameter λ for which the monodromy matrix $V(\omega, \lambda)$ has the eigenvalue 1 can be bifurcation values. The converse assertion is not true in general. We note that a number of important results on bifurcation points were obtained in ⁽⁶⁾.

For simplicity we shall assume that 1 is an eigenvalue of the matrix $V(\omega, 0)$, and we shall be interested in the question of the existence and stability of small periodic solutions for small values of λ .

The case when 1 is a simple eigenvalue of the matrix $V(\omega, 0)$ was studied in detail in ⁽³⁾. In ⁽⁷⁾ the case was studied in which the eigenvalue 1 has order equal to 1 and its multiplicity is k . In this note the general case is considered.

2. Let e_j^i ($i = 1, \dots, k; j = 0, \dots, l_i - 1$) be the eigenvectors and associated vectors of the matrix $V(\omega, 0)$ corresponding to the eigenvalue 1, i.e.

$$V(\omega, 0)e_0^i = e_0^i \quad (i = 1, \dots, k); \quad V(\omega, 0)e_j^i = e_j^i + e_{j-1}^i.$$

Let g_j^i be the eigenvectors and associated vectors of the adjoint matrix, i.e.

$$V^*(\omega, 0)g_0^i = g_0^i \quad (i = 1, \dots, k); \quad V^*(\omega, 0)g_j^i = g_j^i + g_{j-1}^i \quad (j = 1, \dots, l_i - 1).$$

Without loss of generality (see (8, 9)) one may assume that the equalities

$$(e_i^j, g_p^q) = 1 \quad \text{when } j = q, \quad i + p = l_j - 1;$$

$$(e_i^j, g_p^q) = 0 \quad \text{when } j \neq q \text{ or } i + p \neq l_j - 1. \quad (3)$$

Write system (1) in the form

$$dx/dt = A(t, \lambda)x + B_m[t, x, \lambda] + F(t, x, \lambda), \quad (4)$$

where $B_m[t, x, \lambda]$ denotes the homogeneous terms of order m in the spatial variables x , and $F(t, x, \lambda)$ denotes terms of higher order of smallness.

Let $x_{1,0}(t), \dots, x_{k,0}(t)$ denote solutions of the linear system (2) for $\lambda = 0$, satisfying the initial conditions $x_{i,0}(0) = e_0^i$ ($i = 1, \dots, k$). Introduce the homogeneous operator of degree m

$$\Phi_m[\eta] = \left\{ \int_0^\omega \left(V^{-1}(\tau, 0)B_m \left[\tau; \sum_{j=1}^k \eta_j x_{j,0}(\tau); 0 \right], g_0^i \right) d\tau \right\}_{i=1, \dots, k}, \quad (5)$$

where $\eta = (\eta_1, \dots, \eta_k)$.

It can be shown that, in the case when there exists an $\alpha > 0$ for which the inequality

$$\|\Phi_m[\eta]\| \geq \alpha \|\eta\|^m, \quad (6)$$

is satisfied, system (1) for $\lambda = 0$ has no small nonzero ω -periodic solutions.

Next write the matrix $A(t, \lambda)$ in the form

$$A(t, \lambda) = A_0(t) + \lambda^p A_1(t) + o(\lambda^p), \quad (7)$$

where $p > 0$ is some integer. Let

$$B_0 = \left(\int_0^\omega (V^{-1}(\tau, 0)A_1(\tau)x_{i,0}(\tau), g_0^j) d\tau \right)_1^k.$$

Then the following is true.

Theorem 1. Suppose condition (6) is satisfied and the equation

$$\Phi_m[\sigma] + B_0\sigma = 0 \tag{8}$$

has a finite number of real solutions $\sigma_1, \dots, \sigma_q$. Suppose, moreover, that each matrix $m\Phi_m[\sigma_i^{m-1}; \cdot] + B_0$ ($i = 1, \dots, q$) is nonsingular.

Then zero is a bifurcation value of the parameter, and for every sufficiently small value $\lambda > 0$ system (1) has q small nonzero ω -periodic solutions.

The proof uses the well-known Schmidt transformation ^(10,11) and the results of work ⁽¹²⁾.

Equation (8) will be called the characteristic equation. If the matrices $m\Phi_m[\sigma_i^{m-1}; \cdot] + B_0$ are singular, then the roots of the characteristic equation split the small ω -periodic solutions of system (1) into groups, i.e., for each root σ_i of the characteristic equation one can determine a finite or infinite number of small nonzero ω -periodic solutions that depend “essentially” on σ_i . The group of small nonzero ω -periodic solutions corresponding to the root σ_i will be denoted by $K(\sigma_i)$. In connection with these assertions we also note work ⁽¹³⁾.

3. It is said that to the eigenvalue λ_0 of the matrix A_0 there corresponds a Jordan block of multiplicity k and length l , if to the zero eigenvalue of the matrix $A_0 - \lambda I$ there correspond k Jordan blocks of equal length l (see ⁽⁸⁾).

Let σ_i be some root of the characteristic equation. Introduce the matrix $\Delta_i = B_0 - m\Phi_m[\sigma_i^{m-1}; \cdot]$.

Theorem 2. Suppose the monodromy matrix $V(\omega, 0)$ has the eigenvalue 1, to which there corresponds one Jordan block of multiplicity k and length l .

Then, if the matrix Δ_i has at least one nonzero eigenvalue, then for $l \geq 3$ all small ω -periodic solutions from the group $K(\sigma_i)$ are Lyapunov unstable.

Denote by S the linear span of the eigenvectors and generalized eigenvectors of the matrix $V(\omega, 0)$ corresponding to the eigenvalue 1. Suppose that to 1 there correspond r Jordan boxes of lengths l_1, \dots, l_r and multiplicities q_1, \dots, q_r , respectively. Denote the linear span of the q_j eigenvectors and the corresponding generalized eigenvectors by S_j . There is a decomposition $S = S_1 \oplus \dots \oplus S_r$, and

$$\dim S = \sum_{i=1}^r q_i l_i, \quad k = \sum_{i=1}^r q_i.$$

Denote by S_j^0 the linear span of the eigenvectors in S_j , so that $\dim S_j^0 = q_j$. Denote the projector onto S_j^0 by P_j .

Theorem 3. Suppose the matrix $V(\omega, 0)$ has the eigenvalue 1, to which there correspond r Jordan boxes of multiplicities q_1, \dots, q_r and lengths l_1, \dots, l_r , respectively.

Then, if for some j_0 the matrix $P_{j_0} \Delta_i P_{j_0}$ has a nonzero eigenvalue, then for $l_{j_0} \geq 3$ all nonzero small ω -periodic solutions of the group $K(\sigma_j)$ are Lyapunov unstable.

In the case when $l_j \leq 2$ ($j = 1, \dots, r$), the appearance of both stable and unstable small ω -periodic solutions in the group $K(\sigma_j)$ is possible. The corresponding criteria are rather cumbersome. We do not give them here.

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