MATHEMATICS

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Abstract

Full Text

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ON THE LAW OF THE ITERATED LOGARITHM

The present article is devoted to the law of the iterated logarithm for a sequence of independent random variables. In this exposition the latter are represented by measurable elements of a topological semifield. The results obtained are a generalization and refinement of certain theorems available in the literature on the iterated logarithm.

In what follows we shall adhere to the definitions and notation of works ([1,2]). At the same time, for continuity of exposition, we give here a number of definitions and facts concerning topological semifields and their Boolean algebras.

1. Let E be a complete topological semifield, $\overline{K}$ the cone of nonnegative elements in E, and ∇ the topological Boolean algebra of all idempotents from E ([1]).

Definition 1. A subset $I \subset \nabla$ will be called a **confinal** if: 1) from $e \in I$ it follows that $Ce \in I$; 2) from $e, g \in I$ it follows that $e \vee g \in I$, $e \wedge g \in I$.

Definition 2. A nonnegative continuous function m, defined on the confinal I, taking finite or infinite values, finitely additive for disjoint elements of the confinal I and equal to zero only on the idempotent θ (the zero of the algebra ∇), will be called a **measure**.

Further, let $e \in \nabla$. Define the number

$$m^*(e) = \inf_{\substack{a_i \in I \\ \bigvee_{i=1}^{\infty} a_i \geq e}} \sum_{i=1}^{\infty} m(a_i).$$

An element $e (\in \nabla)$ will be called **measurable** if for any g from ∇

$$m^*(g) = m^*(g \wedge e) + m^*(g \wedge Ce).$$

The set of all measurable elements of ∇ will be denoted by I_∇ and I_∇ will be called the **maximal confinal**. The usual properties of a confinal and a measure are given in ([2]).

Definition 3. The triple (E, I_∇, m) will be called a **space with measure**.

Throughout this paper it is assumed that the space with measure is fixed.

Definition 4. An element $x \in E$ is called **measurable** if its support (see ([1])) $a(x) \in I_\nabla$ and, for any real number λ, the relation $g_\lambda = \bigvee\{e : xe < \hat{\lambda} e\} \in I$ holds. Without dwelling on the properties of measurable elements (they coincide with the properties of measurable functions), we introduce the concept of the integral of a simple element. An element $x \in E$ is called **simple** if it has the form:

$$x = \sum_{k=1}^{n} \lambda_k e_k,$$

where $e_1, e_2, \dots, e_n$ are pairwise disjoint measurable elements from I_∇, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are real numbers.

The number

$$\mu(x) = \sum_{k=1}^{\infty} \lambda_k m(e_k)$$

will be called the **integral** of x, if $m(e_k) < \infty$ for $k = 1, 2, \dots, n$; in this case x is called an **integrable simple element**. Just as in the classical theory of integration, this integral can be extended to measurable elements of the semiring E, and its properties are analogous to the classical ones. Elements E for which the integral exists are called **summable**, and their set, as usual, is denoted by L. Obviously, L is a linear topological space in the topology induced from E. An element $x \in L \cap \bar{K}$ will be called a **distribution** if $\mu(x) = 1$.

Introduce the notation $P = \{x : x \in L \cap \bar{K},\ \mu(x) = 1\}$. Each element $x \in P$ generates a continuous probability measure μ_x, defined on I_∇, namely $\mu_x(e) = \mu(xe)$, where $e \in I_\nabla$. The triple (I_∇, m, x) will be called a **probability space**; here $x(\in P)$ is a fixed element (for more details see (2)).

The pair (ξ, x), where $\xi \in E$, $x \in P$, will be called a **random variable**; since x is fixed, we shall simply write ξ. Random variables ξ and η are called **independent** if

$$\mu_x[a((\xi - \alpha 1)_-) a((\eta - \beta 1)_-)] = \mu_x[a((\xi - \alpha 1)_-)] \mu_x[(\eta - \beta 1)_-],$$

where α and β are real numbers, $z_- = (-z \vee \theta)$ is the negative part of the element z, and 1 is the unit (or maximal element) of the Boolean algebra ∇.

Along with the semiring E, introduce for consideration the complex semiring $\mathcal{E}$, defined by the equality $\mathcal{E} = E + iE$. The rules of operation for elements of $\mathcal{E}$ are the same as for complex numbers. Below the Laplace transform is introduced, written in our notation. Its introduction is based on the following assumption.

Let E be a complete semiring ([1]), and let A be a family of power series absolutely convergent on the entire real line. By $\varphi(A)$ we denote the family of series obtained from a series belonging to the family A by replacing the real variable by an element of the semiring E. Then the series from $\varphi(A)$ converge absolutely on E.

2. Consider a sequence $X_1, X_2, \dots, X_n, \dots$ of independent random variables. Everywhere we shall assume that the random variables X_k have zero mathematical expectations and finite variances.

We shall say that the sequence of sums $\{S_n\}$ ($S_n = X_1 + \dots + X_n \in E$) obeys the law of the iterated logarithm (l.i.l.) if

$$\mu_x\left(\bigvee_{e\in I_\triangledown}\left\{e : \bigwedge_{m=1}^{\infty}\bigvee_{n=m}^{\infty}\frac{S_n e}{\chi(\mu(xS_n^2))} = 1\right\}\right) = 1,$$

$$\mu_x\left(\bigvee_{e\in I_\triangledown}\left\{e : \bigvee_{m=1}^{\infty}\bigwedge_{n=m}^{\infty}\frac{S_n e}{\chi(\mu(xS_n^2))} = -1\right\}\right) = 1,$$

where $\chi(t) = \sqrt{2t\ln\ln t}$. Introduce for consideration the idempotent

$$g_k = \bigvee_{g\in I_\triangledown}\{g : |X_k|g \le \tfrac{1}{2}m_k g\}, \tag{1}$$

where

$$m_k = o\left[\left(\frac{\mu(xS_n^2)}{\ln\ln\mu(xS_n^2)}\right)^{1/2}\right].$$

Put

$$\xi_k g_k = (X_k - \mu(xX_k g_k))g_k, \qquad \xi_k\bar{g}_k = -\mu(xX_k g_k)\bar{g}_k;$$

$$\eta_k g_k = \mu(xX_k g_k)g_k; \qquad \eta_k\bar{g}_k = (X_k + \mu(xX_k g_k))\bar{g}_k,$$

where $\bar{g}_k = 1 - g_k$.

It is clear from the construction that $\{\xi_k\}$ and $\{\eta_k\}$ form sequences of independent random variables.

Main theorem. *If, as $n \to \infty$, $\mu(xS_n^2) \to \infty$ and the series*

$$\sum_k \frac{\mu(xX_k\bar{g}_k^\delta)}{\mu(xS_k^2)},$$

where g_k is defined by relation (1), converges, then the law of the iterated logarithm is applicable to the sequence of sums $\{S_n\}$.

Let us note that for $\bar{g}_k = \theta$ ($k = 1, 2, ...$), the well-known theorem of A. N. Kolmogorov ([3]) on the LIL follows from this theorem.

The proof of the first part of the theorem (which is analogous to the well-known proof of A. N. Kolmogorov ([3])) is based on the use of exponential estimates for the sums

$$\xi_n = \sum_{k=1}^n \xi_k.$$

The derivation of these estimates, in turn, rests on the equality

$$\mu\left(xe^{z\xi_n}\right) = \exp\left\{\frac{z^2\mu(x\xi_n^2)}{2}(1+\gamma(z,n))\right\}.$$

Here

$$\gamma(z,n) = \frac{(e^\delta - 1 - \delta)(3 - e^\delta + \delta)m_n^*|z|}{\delta^3(2 - e^\delta + \delta)(1 - m_n^*|z|/\delta)}\,\theta(z),$$

where $|\theta(z)| < 1$, $m_n^* = \max_{k\le n} m_k$, and $0 < \delta < 1$.

Theorem 1. If

$$\lim_{n\to\infty}\frac{1}{n}\mu(xS_n^2) > 0, \qquad \mu(xX_k^2\bar{g}_k) = O\left(\frac{1}{f(k)}\right),$$

where $f(k)$ is some monotone sequence of positive numbers satisfying the condition

$$\sum_{k=1}^\infty \frac{1}{kf(k)} < \infty,$$

then the LIL is applicable to the sequence $\{S_n\}$.

Theorem 2. If

$$\lim_{n\to\infty}\frac{1}{n}\mu(xS_n^2) > 0, \qquad \mu\left(x|X_k|^2\varphi(X_k)\right) < \infty,$$

for all k, and

$$\overline{\lim}_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \mu\left(x|X_k|^2\varphi(X_k)\right) < \infty,$$

then the LIL is applicable to the sequence $\{S_n\}$, where $\varphi(X_k) > 0$ and

$$\sum_{k=1}^\infty \frac{1}{k\varphi(k)} < \infty.$$

The proof of these theorems reduces to verifying the conditions of the main theorem.

From Theorem 2, for $\varphi(X_k) = |X_k|^\delta$ $(\delta > 0)$, one obtains the theorem of V. V. Petrov ([4]). Finally, as in Marcinkiewicz-Zygmund ([5]) and M. Weiss ([6]), it is not difficult to construct examples such that if, in our condition (1), o is replaced by O, then the LIL will not hold.

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Note: Figure translations are in progress. See original paper for figures.