

# ONE VARIATIONAL PROBLEM IN PARAMETRIC FORM

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**Abstract**

**Full Text**

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*MATHEMATICS*

Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

## ONE VARIATIONAL PROBLEM IN PARAMETRIC FORM

Let  $H(x_1, x_2, x_3)$  be a positive homogeneous function of the first degree that is strictly convex. This means that for any  $\lambda \geq 0$

$$H(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda H(x_1, x_2, x_3)$$

and the second differential  $d^2H$  is a positive definite form.

We consider the following variational problem. Find a regular surface  $F : r = r(u, v)$  with a given spherical image  $\omega$ , with a given support function along the boundary ( $rn = \psi$ ), which gives the functional

$$I(F) = \iint_F H(A_1, A_2, A_3) du dv$$

a stationary value ( $\delta I = 0$ ). Here  $A_1, A_2, A_3$  are the components of the vector product  $r_u \times r_v$ . By a regular surface we mean the image of an arbitrary domain of the  $uv$ -plane by means of a regular vector function  $r(u, v)$ . The usual requirement  $r_u \times r_v \neq 0$  is not assumed, since geometric singularities on the surface are not excluded.

**Theorem.** *Whatever the function  $H$  of class  $C^{2,\lambda}$ , the domain  $\omega$  of class  $C^{2,\lambda}$  lying entirely on one hemisphere of the unit sphere, and the function  $\psi$ , prescribed on the boundary  $\gamma$  of the domain  $\omega$ , the variational problem stated above is always solvable in the class of surfaces  $C^{2,\lambda}$ .*

**Proof.** We shall seek the solution of our variational problem among surfaces  $F$  with nonpositive Gaussian curvature and one-to-one spherical image. Such a surface can be specified by its support function  $\Phi(n) = rn$ , where  $r$  is the vector of a point of the surface and  $n$  is the unit normal vector at this point.

By the condition of the theorem, the domain  $\omega$  (the spherical image of the sought surface) lies entirely on one hemisphere. Without loss of generality, one may assume that this hemisphere is

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_3 > 0.$$

The domain  $\omega$  is projected from the origin of coordinates onto a certain domain  $\bar{\omega}$  in the plane  $x_3 = 1$ , bounded by a curve  $\bar{\gamma}$  of class  $C^{2,\lambda}$ . To each point  $n$  of the domain  $\omega$  under this projection there corresponds a certain point  $(x_1, x_2)$  of the domain  $\bar{\omega}$  in the plane  $x_3 = 1$ . In our assumption the numbers  $x_1, x_2$  may be regarded as coordinates on the surface  $F$ .

By virtue of the homogeneity of the function  $H$ , we have

$$H(A_1, A_2, A_3) du dv = H(n_1, n_2, n_3) |r_u \times r_v| du dv = H(n_1, n_2, n_3) d\sigma,$$

where  $n_1, n_2, n_3$  are the components of the vector  $n$  (the unit normal to the surface  $F$ ), and  $d\sigma$  is the area element of the surface  $F$ .

Put  $\Phi(x) = (rx)$ , where  $x$  is any nonzero vector directed into the domain  $\omega$ , and  $r$  is the vector of a point of the surface  $F$  at which the normal ( $n$ ) has the direction of the vector  $x$ . The function  $\Phi(x)$  is positively homogeneous of the first degree. The coordinates of points of the surface  $F$  are expressed through the derivatives of the function  $\Phi$ . Namely,

$$x'_1 = \partial\Phi/\partial x_1, \quad x'_2 = \partial\Phi/\partial x_2, \quad x'_3 = \partial\Phi/\partial x_3.$$

Setting  $\Phi(x_1, x_2, 1) = \varphi(x_1, x_2)$ , we obtain

$$x'_1 = \varphi_1, \quad x'_2 = \varphi_2, \quad x'_3 = \varphi - x_1\varphi_2 - x_2\varphi_1,$$

where  $\varphi_1$  and  $\varphi_2$  are the first derivatives of the function  $\varphi$  with respect to the variables  $x_1$  and  $x_2$ , respectively. With the aid of these formulas for the coordinates of points of the surface  $F$ , we find the expression for the area element

$$d\sigma = -(\varphi_{11}\varphi_{22} - \varphi_{12}^2)\sqrt{1 + x_1^2 + x_2^2},$$

where  $\varphi_{ij}$  are the second derivatives of the function  $\varphi$ .

Noting that

$$n_1 = \frac{x_1}{\sqrt{1 + x_1^2 + x_2^2}}, \quad n_2 = \frac{x_2}{\sqrt{1 + x_1^2 + x_2^2}}, \quad n_3 = \frac{1}{\sqrt{1 + x_1^2 + x_2^2}},$$

we obtain

$$I(F) = - \iint_{\bar{\omega}} h(x_1, x_2) (\varphi_{11}\varphi_{22} - \varphi_{12}^2) dx_1 dx_2,$$

where, for brevity, we have denoted  $H(x_1, x_2, 1) \equiv h(x_1, x_2)$ .

Let us form the Euler equation for the functional  $I$ . We obtain

$$h_{22}\varphi_{11} - 2h_{12}\varphi_{12} + h_{11}\varphi_{22} = 0. \quad (*)$$

From the positive definiteness of the form  $d^2H$  follows the positive definiteness of the form  $d^2h$  and, consequently, the ellipticity of the equation for the function  $\varphi$ .

The finding of the surface  $F$  solving our variational problem is reduced to solving equation (\*) in the domain  $\bar{\omega}$  with the boundary condition

$$\varphi|_{\gamma} = \psi\sqrt{1 + x_1^2 + x_2^2}.$$

As is known, this boundary-value problem is always solvable in the class  $C^{2,\lambda}$ , if the coefficients of the equation  $(h_{ij})$  belong to  $C^{0,\lambda}$ , and the curve  $\gamma$  bounding the domain is of class  $C^{2,\lambda}$  (1). In our case these conditions are satisfied.

If  $\varphi(x_1, x_2)$  is a solution of the boundary-value problem for equation (\*), then the surface  $F$  solving our variational problem is given by the equations

$$x'_1 = \partial\varphi/\partial x_1, \quad x'_2 = \partial\varphi/\partial x_2, \quad x'_3 = \varphi - x_1\partial\varphi/\partial x_1 - x_2\partial\varphi/\partial x_2.$$

We note that the curvature of the surface  $F$  has the sign of the discriminant  $\varphi_{11}\varphi_{22} - \varphi_{12}^2$  and, consequently, by virtue of the ellipticity of equation (\*), is nonpositive, as was assumed. The theorem is proved.

Physical-Technical Institute  
of Low Temperatures  
Academy of Sciences of the Ukrainian SSR

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## CITED LITERATURE

<sup>1</sup> K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.

*Note: Figure translations are in progress. See original paper for figures.*

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