APPROXIMATE DIFFERENTIATION BY MEANS OF LAGRANGE AND HERMITE INTERPOLATION POLYNOMIALS

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Abstract

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N. S. BAIGUZOV

APPROXIMATE DIFFERENTIATION BY MEANS OF LAGRANGE AND HERMITE INTERPOLATION POLYNOMIALS

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As is well known, one of the methods for the approximate computation of a derivative is to replace it by the differentiated Lagrange interpolation polynomial ([1]-[3]). In connection with this, for a function $f(x)$ continuously differentiable on the segment $[-1, 1]$, consider the quantity

$$R_n'(f, x, M) = f'(x) - L_n'(f, x, M), \tag{1}$$

where

$$L_n(f, x, M) = \sum_0^n f(x_k^{(n+1)})\, l_{k,n+1}(x), \tag{2}$$

$$l_{k,n+1}(x) = \frac{\omega_{n+1}(x)}{\omega_{n+1}'(x_k^{(n+1)})(x - x_k^{(n+1)})}, \qquad \omega_{n+1}(x) = \prod_0^n (x - x_k^{(n+1)}), \tag{3}$$

and $M = \|x_k^{(n+1)}\|$ is a certain infinite triangular matrix of nodes located on the segment $[-1, 1]$. In papers ([1], [4], [6]) cases are considered in which the quantity (1) is estimated in terms of the derivative of order n of the function $f(x)$.

In the present paper estimates are given for the rate of decrease of the quantity (1) under the condition that the function $f(x)$ is only continuously differentiable on $[-1, 1]$, and the matrix of nodes is defined by one of the formulas

$$x_0^{(n+2)} = 1, \qquad x_{n+1}^{(n+2)} = -1, \qquad x_k^{(n+2)} = \cos\frac{2k-1}{2n}\pi \quad (k = 1, \dots, n); \tag{4}$$

$$x_k^{(n+1)} = \cos k\frac{\pi}{n} \quad (k = 0, \dots, n). \tag{5}$$

From (3) and (4) it follows that $\omega_{n+2}(x) = (x^2-1)\widetilde{T}_n(x)$, where $\widetilde{T}_n(x)$ is the Chebyshev polynomial of the first kind with leading coefficient equal to one. Similarly, under condition (5) we have $\omega_{n+1}(x) = (x^2-1)\widetilde{U}_{n-1}(x)$, where $\widetilde{U}_{n-1}(x)$ is the Chebyshev polynomial of the second kind.

Let us denote the matrices of nodes (4) and (5), respectively, by M_1 and M_2, and put

$$D_n(x, M_1) = \sum_1^n |l'_{k,n+2}(x)| \sin\theta_k^{(n+2)}, \tag{6}$$

$$D_n(M_1) = \max_{x\in[-1,1]} D_n(x, M_1). \tag{7}$$

Lemma 1. If the matrix of nodes is defined by formula (4), then the inequalities hold

$$\frac{2}{\pi} n \ln \frac{1}{\sin \pi/n} < D_n(M_1) < \frac{26}{\pi} n \ln n + O(n),$$

$$c_1 n \leq \max_{x\in\Delta_n} D_n(x, M_1) \leq c_2 n,$$

where the set Δ_n consists of the points $x_k^{(n)} = \cos k\pi/n$ $(k = 1, \dots, n-1)$ and the zeros of the derivative of the polynomial $\omega_{n+2}(x) = (x^2-1)\widetilde{T}_n(x)$.

The proofs of these inequalities are obtained with the aid of formulas (2) and (3), if one takes into account that, by virtue of (4), here we have $\omega_{n+2}(x) = (x^2-1)\pi_n(x)$. Analogous results also hold for the matrix M_2.

Theorem 1. If the function $f(x)$ is continuously differentiable on the segment $[-1,1]$ and $\omega(\delta, f')$ is the modulus of continuity of its derivative $f'(x)$, then for the Lagrange interpolation polynomial corresponding to the matrix M_1 the inequalities

$$\left|R'_{n+1}(f, x, M_1)\right| \leq c_3 \omega(1/n, f') \ln n, \qquad x \in [-1,1], \tag{8}$$

$$\left|R'_{n+1}(f, x, M_1)\right| \leq c_4 \omega(1/n, f'), \qquad x \in \Delta_n \tag{9}$$

hold.

Under the conditions of Theorem 1, in paper ([7]) the existence is proved of polynomials $\{Q_n(x)\}$ for which, for $k = 0, 1$, the estimates

$$\left|f^{(k)}(x) - Q_n^{(k)}(x)\right| \leq c_5 \frac{\sqrt{1-x^2}^{\,1-k}}{n^{1-k}}\, \omega\left(\frac{\sqrt{1-x^2}}{n}, f'\right), \qquad x \in [-1,1].$$

Using these polynomials in the inequality

$$\left|f'(x) - L'_{n+1}(f, x, M_1)\right| \le |f'(x) - Q'_n(x)| + \sum_1^n \left|f\left(x_k^{(n+2)}\right) - Q_n\left(x_k^{(n+2)}\right)\right| \left|l'_{k,n+2}(x)\right|,$$

with the aid of Lemma 1 we obtain the estimates (8) and (9).

It is not hard to prove that estimate (9) is sharp in order. Indeed, if, for example, $f_1(x) = -x^2(1-x^2)$ for $-1 \le x < 0$ and $f_1(x) = x^2(1-x^2)$ for $0 \le x \le 1$, then we shall have $f'_1(x) \in \mathrm{Lip}\,1$ and $f'_1(0) = 0$. Therefore, for $n = 2p$ we find

$$L'_{n+1}(f_1, 0, M_1) = -\frac{1}{n\cos\pi/2n}.$$

If the function $f(x)$ has p continuous derivatives on the segment $[-1,1]$, with the p-th derivative having modulus of continuity $\omega(\delta, f^{(p)})$, then in the case of the matrix M_1 the estimates

$$\left|R'_{n+1}(f, x, M_1)\right| \le c_6 \frac{\ln n}{n^{p-1}}\, \omega\left(\frac{1}{n}, f^{(p)}\right), \qquad x \in [-1,1], \tag{10}$$

$$\left|R'_{n+1}(f, x, M_1)\right| \le c_7 \frac{1}{n^{p-1}}\, \omega\left(\frac{1}{n}, f^{(p)}\right), \qquad x \in \Delta_n \tag{11}$$

are valid.

Analogous facts also hold for the matrix M_2. Further, one may consider a matrix M for which the conditions

$$0 < c_8 \le \max_{[x_k^{(n)}, x_{k+1}^{(n)}]} |\omega_n(x)| \Big/ \max_{[x_i^{(n)}, x_{i+1}^{(n)}]} |\omega_n(x)| \le c_9 < \infty,$$

are fulfilled, where $\omega_n(x)$ is defined by formula (3) for the matrix M, and $x_0^{(n)} = 1$, $x_{n+1}^{(n)} = -1$. The convergence of the interpolation process for such matrices is considered in papers ([8,9]). It is not hard to show that estimates of the form (8), (10) also hold here.

Let us now consider the quantity

$$r_{2n+1}(f, x, M) = f'(x) - H'_{2n+1}(f, x, M), \tag{12}$$

where

$$H_{2n+1}(f, x, M) = \sum_0^n f\left(x_k^{(n+1)}\right) U_{k,n+1}(x) + \sum_0^n f'\left(x_k^{(n+1)}\right) V_{k,n+1}(x), \tag{13}$$

$$U_{k,n+1}(x) = \left[1 - \frac{\omega''_{n+1}\left(x_k^{(n+1)}\right)}{\omega'_{n+1}\left(x_k^{(n+1)}\right)}\left(x - x_k^{(n+1)}\right)\right] l^2_{k,n+1}(x), \tag{14}$$

$$V_{k,n+1}(x) = \left(x - x_k^{(n+1)}\right) l^2_{k,n+1}(x). \tag{15}$$

Similarly to (6) and (7), here we set

$$B_{n+1}(f,x,\alpha)=\sum_{0}^{n}|f_k'(x)|\sin^\alpha\theta_k^{(n+1)},\qquad \alpha\geqslant 0, \tag{16}$$

$$B_{n+1}(f,E,\alpha)=\max_{x\in E}B_{n+1}(f,x,\alpha),\qquad E=[-1,1]. \tag{17}$$

Lemma 2. If the matrix of nodes is determined by formula (4), then for $x\in[-1,1]$ the inequality

$$B_{n+1}(V,x,1)\leqslant\frac{2}{n}|\omega_{n+2}'(x)|\left\{\frac{4}{\pi}|\widetilde{T}_n'(x)|\ln n+O(1)\right\}+O(1).$$

holds.

Lemma 3. If the matrix of nodes is M_1, then the inequalities

$$c_{10}n\ln n\leqslant B_{n+1}(U,[-1,1],2)\leqslant c_{11}n\ln n,$$

$$c_{12}n\leqslant B_{n+1}(U,\Delta_n)\leqslant c_{13}n.$$

hold.

Theorem 2. If the matrix of nodes is M_1 and the function $f(x)$ is twice continuously differentiable on the segment $[-1,1]$, then the estimates

$$|r_{2n+3}'(f,x,M_1)|\leqslant c_{14}\frac{\ln n}{n}\,\omega\left(\frac{1}{n},f''\right),\qquad x\in[-1,1], \tag{18}$$

$$|r_{2n+3}'(f,x,M_1)|\leqslant c_{15}\frac{1}{n}\,\omega\left(\frac{1}{n},f''\right),\qquad x\in\Delta_n. \tag{19}$$

hold.

Theorem 3. If the matrix of nodes is M_2 and the function $f(x)$ is continuously differentiable on the segment $[-1,1]$, then the inequalities

$$|r_{2n+1}'(f,x,M_2)|\leqslant c_{16}\omega\left(\frac{1}{n},f'\right)\ln n,\qquad x\in[-1,1], \tag{20}$$

$$|r_{2n+1}'(f,x,M_2)|\leqslant c_{17}\omega\left(\frac{1}{n},f'\right),\qquad x\in\Delta_n', \tag{21}$$

hold, where Δ'_n is the set of zeros of the derivative of the polynomial $\omega_{n+1}(x) = (x^2 - 1)U_{n-1}(x)$.

As examples show, inequalities (19) and (21) are exact in order. If the function $f(x)$ has p continuous derivatives, then the estimates (18)–(21) are modified analogously to (10)–(11).

One can consider analogous questions for the matrices M_1^* and M_2^*, obtained respectively from the matrices M_1 and M_2 by deleting the end points $x = \pm 1$.

Theorem 4. If the matrix of nodes is M_1^* and the function $f(x)$ is twice continuously differentiable on the segment $[-1, 1]$, then the estimates

$$|R'_{n-1}(f, x, M_1^*)| \leqslant c_{18}\omega\left(\frac{1}{n}, f''\right)\ln n, \qquad x \in [-1, 1],$$

$$|R'_{n-1}(f, x, M_1^*)| \leqslant c_{19}\frac{1}{n}\,\omega\left(\frac{1}{n}, f''\right), \qquad x \in \Delta''_n,$$

hold, where Δ''_n is the set of zeros of the derivative of the polynomial $\omega_n(x) = \widetilde{T}_n(x)$.

Theorem 5. If the matrix of nodes is M_1^* and the function $f(x)$ is continuously differentiable on the segment $[-1, 1]$, then the estimates

$$|R'_{n-1}(f, x, M_1^*)| \leqslant c_{20}\frac{1}{\sqrt{1 - x^2}}\,\omega\left(\frac{1}{n}, f'\right)\ln n, \qquad x \in (-1, 1)$$

$$|R'_{n-1}(f, x, M_1^*)| \leqslant c_{21}\omega\left(\frac{1}{n}, f'\right), \qquad x \in \Delta''_n.$$

Estimates of the same form are also valid for the case of Hermite interpolation. For example, if $f'(x) \in \mathrm{Lip}\,1$ and $x \in \Delta''_n$, then the estimate

$$|r'_{2n-1}(f, x, M_1)| \leqslant c_{22}/n, \qquad x \in \Delta''_n$$

holds.

On the other hand, if one sets $f_2(x) = -x^2/2$ for $-1 \le x < 0$ and $f_2(x) = x^2/2$ for $0 \le x \le 1$, then $f'_2(x) \in \mathrm{Lip}\,1$, $f'_2(0) = 0$, and for $n = 2p$ we have

$$H'_{2n-1}(f_2, 0, M_1^*) = -\frac{1}{2}\,\frac{1}{n^2 \sin \pi/2n}.$$

In conclusion, we note some results on the approximate differentiation of periodic functions. Denote by m_1^* the matrix of nodes

$$x_k^{(n)} = \frac{2n - 2k + 1}{2n}\pi \quad (k = 1, \dots, 2n),$$

and let there correspond to it the trigonometric interpolation polynomial $t_{n-1}(f, x, m_1^*)$.

Theorem 6. If an even periodic function $f(x)$ is continuously differentiable p times, and $E_n(f^{(p)})$ is the best approximation of $f^{(p)}(x)$ by trigonometric polynomials of degree not exceeding n, then the estimate

$$\left| f^{(p)}(x) - t_{n-1}^{(p)}(f, x, m_1^*) \right| \le (c_{23} + c_{24} \ln n)\, E_n(f^{(p)}).$$

holds.

In particular, if $p = 1$, then the first estimate of Theorem 5 follows from this. Analogous results are obtained also in the case where the matrices M_1 and M_2 are transformed in the corresponding way.

In all the estimates given above, certain (not best possible) values of the constants have been computed.

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Ural State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.