STRUCTURAL PROPERTIES OF SOME DEFINING FUNCTIONALS

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Abstract

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STRUCTURAL PROPERTIES OF SOME DEFINING FUNCTIONALS

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It was proved in ([1]) that a segment-functional of the form

$$(\mu_i)_0^n = 0_0, \dots, 0_{s-1}, 1_s, \theta_1, \dots, \theta_l \qquad (s + l = n;\ s > n/2 + 1) \qquad (*_s^+)$$

determines, for the variables (θ_i) in a bounded l-dimensional domain, all extremal polynomials $\{Q_n(x)\}$ of class II $[0, s, 0]$, and only them.

Since at each point $(\theta_i)_1^l$ of the infinite space the segment $(*_s^+)$ is also served by a unique extremal polynomial (only of another passport) ([1]), it is of interest to determine the capacity of such service in view of the fact that segments of type $(*_s^+)$ are convenient for the analytic construction of extremal polynomials ([1]), and also to generalize the form $(*_s^+)$.

Recall the general criterion of extremality ([1]): $Q_n(x) = \sum_{i=0}^n q_i x^i$ serves $(\mu_i)_0^n$ if and only if it is reduced, i.e.

$$\max_{[0,1]} |Q_n(x)| = 1,$$

and, for its distribution $\overset{+}{\sigma}_i$ (i.e. for the points $0 \le \sigma_1 < \dots < \sigma_s \le 1$, where $Q_n \overset{+}{\sigma} = +1;\ Q_n(\sigma) = -1$), the system of linear equations

$$\sum_{i=1}^s \delta_i \sigma_i^k = \mu_k \qquad (k = 0, 1, \dots, n) \tag{V}$$

gives, for the loads (δ_i), solutions with $\operatorname{sgn} \delta_i = Q_n(\sigma_i)$ or $\delta_i = 0$. If $\delta_k \neq 0$, we shall call σ_k an active node of $Q_n(x)$. The totality of active nodes forms a subdivision, or selection, for $Q_n(x)$ and is the true (unique) distribution of the segment-functional $(\mu_i)_0^n$. The whole system (V) is compatible.

Introduce the following abbreviated notation. If some segment (a) is served by the polynomial $Q_n(x)$, then the segment multiplied termwise by $\alpha > 0$, i.e. $\alpha \cdot (a)$, is also served. If another segment (b) has the same $Q_n(x)$, then the segment

$$(c) = \frac{\alpha \cdot (a) + \beta \cdot (b)}{\alpha + \beta}$$

also has it ($\alpha > 0$; $\beta > 0$); moreover, if (a) and (b) have the form $(*_s^+)$, then so does (c). If (δ_i') and (δ_i'') are the loads of (a) and (b), then the loads of (c) are

$$\frac{\alpha \cdot \delta_i' + \beta \cdot \delta_i''}{\alpha + \beta}.$$

Hence follows

Remark 1. The domain of service of $(*_s^+)$ by a certain selection $(\sigma_i)_1^{s_1}$ of the polynomial $Q_n(x)$ forms a simply connected (convex) set.

In what follows we shall consider segments $(\nu_i)_0^n$ in which s $(1 \leq s \leq n)$ parameters at any prescribed places are equal to zero. Among them there is always $\nu_0 = 0$. The remaining parameters $(\theta_k)_1^{i+n-s}$ are variable, with the exception of $\theta_{k_1} = 1$ or -1 $(k_1 < k_2 < \cdots < k_{n+1-s})$. We shall denote such segments by $\nu_{(s)}^+$ and $\nu_{(s)}^-$, respectively. Separating out all their constant parameters, we obtain the following "bases" :

$$0_0, 0_{l_1}, \ldots, 1_{k_1}, \ldots, 0_{l_{s-1}}, \tag{1^+}$$

$$0_0, 0_{l_1}, \ldots, -1_{k_1}, \ldots, 0_{l_{s-1}} \quad (k_1 \leq s). \tag{1^-}$$

Remark 2. The bases $(1\pm)$ require that the number s_1 of nodes in the segments $(\nu_{(s)}^\pm)$ be $s_1 \geq s+1$; when $s_1 = s+1$, the corresponding loads $(\delta_i)_1^{s+}$ give alternants of the form $\ldots + - +$ for $\nu_{(s)}^+$ and $(\ldots - + -)$ for $(\nu_{(s)}^-)$. This is obvious if, from system (V), one selects $s+1$ equations with right-hand side corresponding, taking account of the numbering, to the numbers $(1\pm)$. The determinant of the selected system is > 0.

Theorem 1. Let $(\nu_{(s)}^+)$ be served at the points $(\theta_{k_i}^{(0)})_{i=1}^{n+1-s}$ by a sample $(\sigma_i)_1^{s_1}$ of the polynomial $Q_n(x)$, with all active nodes and with loads $(\delta_i)_1^{s_1}$, where $s+1 \leq s_1 < S$ (the total number of nodes of $Q_n(x)$). Then, when any σ^* from $Q_n(x)$ is added to the sample, there will always be found a point $(\theta_{k_i}^{(1)})_{i=1}^{n+1-s}$ at which $\nu_{(s)}^+$ is served by the entire sample supplemented by the node σ^*.

Indeed, extend the basis $(1+)$ by zeros, $s_1 - s$ in number, at arbitrary free numbers; we obtain an incomplete segment $(^1)$ with $s_1 + 1$ parameters (zeros and ones). Decompose it with respect to the nodes $(\sigma_i)_1^{s_1}$ with the adjoined σ^*

(it remains unnumbered). We obtain loads $(\Delta_i)_1^{s_1}$ and Δ^* for (σ^*), of alternating signs in the order of increase of all σ, according to Remark 2. Next we fill this segment structurally by these nodes. This means: construct the segment $(\varepsilon_i)_0^n$, in which the same extended basis is present, and at any free place with number p put

$$\varepsilon_p = \sum_{i=1}^{s} \Delta_i \sigma_i^p + \Delta^* \sigma^{*p}.$$

Compare $(\Delta_i)_1^{s_1}$ and $(\delta_i)_1^{s_1}$. There are no zeros among them. Find $a_i > 0$ such that $a_i|\Delta_i| = |\delta_i|$. Choose $0 < \alpha < \min(a_i)$; then $\alpha|\Delta_i| < |\delta_i|$. If $\operatorname{sgn}\Delta^* = Q_n(\sigma^*)$, construct the segment

$$\frac{\nu_s^+ + \alpha(\varepsilon_i)_0^n}{1+\alpha}$$

with the same basis as (ν_s^+), and with loads

$$\frac{\alpha\Delta_i + \delta_i}{1+\alpha} \quad \text{and} \quad \frac{\alpha\Delta^*}{1+\alpha}.$$

Since $\operatorname{sgn}(\alpha\Delta_i + \delta_i) = \operatorname{sgn}\delta_i$, $Q_n(x)$ is its extremal. If $\operatorname{sgn}\Delta^* = -Q_n(\sigma^*)$, require additionally in the choice of α that $\alpha < 1$, and construct the segment

$$\frac{\nu_{(s)}^+ - \alpha(\varepsilon_i)_0^n}{1-\alpha}$$

also with the same basis and with loads

$$\frac{\delta_i - \alpha\Delta_i}{1-\alpha}$$

and

$$-\frac{\alpha\Delta^*}{1-\alpha},$$

i.e. $Q_n(x)$ is its extremal.

Corollary. There will always be found points $(\theta_{k_i})_{i=1}^{n+1-s}$ at which $(\nu_{(s)}^+)$ is served by the original sample of nodes of $Q_n(x)$ with the addition of any number of nodes of $Q_n(x)$ not included in it.

Remark 3. A sample $(\sigma_i)_1^{s_1}$ containing exactly $q = s$ alternants contains, in any $s+1$ of its points with a continuous alternant, only either a $(+)$-alternant or a $(-)$-alternant. If, however, in the sample $q > s$, then it contains both $s+1$ points with a $(+)$-alternant and $s+1$ points with a $(-)$-alternant.

Remark 4. Every sample of nodes of $Q_n(x)$ consisting of $s+1$ points with a $(+)$-alternant serves only either $(\nu_{(s)}^+)$ or $(\nu_{(s)}^-)$, and moreover at only one point. Let $(\sigma_i)_1^{s+1}$ be this sample. The basis $(\nu_{(s)}^+)$, when decomposed with respect to these nodes, gives loads (δ_i), $\operatorname{sgn}\delta_{s+1} = (-1)^{s+k}$, and the structural completion of the basis with respect to these nodes is unique.

Theorem 2. A subsample $(\sigma_i)_1^{s_1}$ of the polynomial $Q_n(x)$, where $s+1 \leq s_1 \leq S$, has the following properties: 1) if the number of alternants $q = s$, then the sample serves one and only one of the two segments of the form $(\nu_{(s)}^{+})$ or $(\nu_{(s)}^{-})$; 2) if $q < s$, the sample serves neither the one nor the other.

The proof is by induction.

Let $s_1 = s + 1$. According to Remark 4, under the condition of a continuous alternant the sample serves one and only one of the segments of the form

$(v_{(s)}^{\pm})$ (and only at one point). Suppose the theorem has been proved for any sample with number of nodes $\leq s_1$; we shall prove its validity for $(\sigma_i)_1^{s_1+1}$ with number of alternations $q \leq s_1$.

1. Let $q = s_1$, and, for definiteness, suppose that this is a $(+)$-alternation (see Remark 3). Then one of the two intervals $(v_{(s)}^{\pm})$, at some point $(\theta_{k_i}^{1})_{i=1}^{n+1-s}$, in accordance with Theorem 1, is served by the sample $(\sigma_i)_1^{\pm,s_1+1}$ with all active nodes and with loads (δ_i'). Let this be an interval of the form $(v_{(s)}^{+})$. Suppose now that $(\sigma_i)_1^{\pm,s_1+1}$ also serves $(v_{(s)}^{-})$ at some point $(\theta_{k_i}')_{i=1}^{n+1-s}$ with all active nodes and loads (δ_i''). Then one must have $\operatorname{sgn}\delta_i' = \operatorname{sgn}\delta_i''$. In the equalities $a_i\delta_i' = \delta_i''$ $(a_i > 0)$, take $\alpha = \min(a_i) = a_p$; then $\alpha\delta_p' = \delta_p''$ and $\alpha|\delta_i'| < |\delta_i''|$ for $i \neq p$. Form the interval

$$(\mu_i)_0^n = \frac{(v_{(s)}^{-}) - \alpha_p(v_{(s)}^{+})}{1+\alpha},$$

it has the nodes (σ_i) for $i \neq p$, with loads

$$\frac{\delta_i'' - \alpha_p\delta_i'}{1+\alpha}$$

in number $\leq s_1$, while the number of alternations $q' \leq s$. By the induction hypotheses, $q' < s$ is impossible. If, however, $q' = s$, then a $(+)$-alternation has been preserved, and then $(\mu_i)_0^n$ of type $(v_{(s)}^{-})$ is served by the same sample as is $(v_{(s)}^{+})$; this too is impossible by the induction hypothesis.

2. Let in $(\sigma_i)_1^{\pm,s_1+1}$ we have $q < s$. Suppose that at the point $(\theta_{k_i}^{(0)})_{i=1}^{n+1-s}$ the sample serves $(v_{(s)}^{+})$. From the general system (V), which gives the structural decomposition of $(v_{(s)}^{+})$ by the nodes $(\sigma_i)_1^{s_1+1}$, choose $s_1 + 1$ equations with s_1+1 unknowns (Δ_i). In this system take all $s+1$ equations with right-hand side equal to the $s + 1$ basis parameters. We shall vary one of the parameters $\theta_l^{(0)}$ of this system by setting $\theta_l^{(0)} = \theta$. Then all Δ_i become linear functions of θ. Let us find, closest to $\theta_l^{(0)} = \theta$, a value of θ at which at least one of the loads Δ_i becomes zero (the others retain their initial signs). Let this occur at $\theta = \theta^{*}$. Then the interval $(v_{(s)}^{+})$, in which

$\theta_l^{(0)}$ is replaced by θ^*, is served by $(\sigma_i)^{\pm}$ with one (at least) node removed, which does not increase the number of alternations; this is impossible by the induction hypothesis. Theorem 2 is proved.

For simplicity of the subsequent exposition, we shall consider only special intervals of the form $0_0, \dots, 0_{s-1}, \pm 1_s, \theta_1, \dots, \theta_{n-s} = (\mu^{\pm}_{(s)})$. All results will also be valid in the general case. We shall compare $(\mu^{\pm}_{(s)})$, $(\mu^{\pm}_{(s+1)})$, and $(\mu^{\pm}_{(s-1)})$. If in $(\mu^{\pm}_{(s)})$ the (θ_i) are variables and range over a domain of the form $\theta_i = \lambda\theta_i' + \theta_i''$, where $\lambda \to \infty$ and $\theta_i' \neq 0$, we shall call such a domain a ray (the simplest cone).

Theorem 3. If $Q_n(x)$ has a subdistribution $(\sigma_i)_1^{\pm, s_1}$ containing a number of alternations $q \geq s+1$, then this sample serves an interval of the form $(\mu^{\pm}_{(s)})$ in an infinite domain of the (θ_i) of ray type.

Take from the sample two groups of $s+1$ points (they may have common nodes), one forming a $(+)$-alternation, the other a $(-)$-alternation. Find the decompositions of the two bases $0_0, \dots, 0_{s-1}, \pm 1$ respectively by these groups of points and, respectively, continue them in a structural way. We have:

$$0_0, \dots, 0_{s-1}, 1, \theta_1^{(1)}, \dots, \theta_{n-s}; \tag{2}$$

$$0_0, \dots, 0_{s-1}, -1, \theta_1^{(2)}, \dots, \theta_{n-s}^{(2)}; \tag{3}$$

$Q_n(x)$ is extremal for (2) and for (3). On the basis of Theorem 2 the original subdistribution also serves some interval

$$0_0, \dots, 0_s, 1_{s+1}, \theta_2^{(0)}, \dots, \theta_{n-s}^{(0)}. \tag{4}$$

Compose $\lambda(4) + (2)$ (or $\lambda \cdot (4) + (3)$), where $\lambda > 0$ is arbitrary. We obtain

$$0_0, \dots, 0_{s-1}, 1, \lambda + \theta_1^{(1)}, \dots, \lambda\theta_{n-s}^{(0)} + \theta_{n-s}^{(1)},$$

and the required ray has been found, if all $\theta_i^0 \neq 0$; but this can always be achieved by a small change of the parameters in the system of equations (V).

Corollary. If $Q_n(x)$ serves (μ_s^+) in a finite domain (θ_i), then it does not serve at all a segment of the form $(\mu_{(s+1)})$ (i.e. one in which the number of alternations is $q = s$).

Theorem 4. If a sample $(\sigma_i)_1^{s_1}$ serves the segment $(\mu^+_{(s)})$ in an infinite domain, then this domain contains a ray.

Let us note that for $s_1 = s+1$ the sample can serve $(\mu^+_{(s)})$ only at one point. Let $s_1 = s+2$, and let the sample serve $(\mu^+_{(s)})$ in an infinite (one-dimensional!) domain. At any point of this domain the loads $(\Delta_i)_1^{s+2}$ are determined from the first $s+2$ equations (V), and therefore are linear functions of one parameter θ_1, i.e. $\Delta_i = \alpha_i \theta_i + \beta_i$ ($\alpha_i \neq 0$); the remaining

$$\theta_k = \sum_{i=1}^{s+2} \Delta_i \sigma_i^{k+s} = \theta_1 \sum_{i=1}^{s+2} \alpha_i \sigma_i^{s+k} + \sum_{i=1}^{s+2} \beta_i \sigma_i^{s+k}.$$

Consequently, θ_1 assumes arbitrarily large values (according to what was noted above, one may assume $\theta_k \neq 0$). Then finally we obtain a functional of the form

$$0_0, \dots, 0_{s-1}, 1_s, \theta_1, \gamma_2\theta_1 + \beta_2, \dots, \gamma_{n-s}\theta_1 + \beta_{n-s}, \quad \gamma_i \neq 0, \tag{5}$$

i.e. a ray. These conclusions remain valid also for $s_1 > s+2$, since the domain (θ_i) served by a poorer subdistribution $Q_n(x)$ is a boundary domain for the domain served by an enriched subdistribution (see Theorem 1).

Corollary 1. The sample of Theorem 4 also serves $(\mu_{(s+1)})$.

Indeed, divide (5) by $|\theta_1|$, and in the limit as $\theta_1 \to \infty$ we obtain

$$0_0, \dots, 0_{s-1}, 0_s, \gamma_2, \dots, \gamma_{n-s}.$$

The mentioned sample, serving $(\mu_{(s+1)})$, must contain not fewer than $s+1$ alternations.

Corollary 2. A subdistribution $Q_n(x)$ containing the number of alternations $q_{\max} = s$ serves $(\mu_{(s)})$ in a finite domain, since otherwise there would, in the contrary case, be a ray in this domain, and then the subdistribution would serve $(\mu_{(s+1)})$, which is impossible.

Remark. The theorems proved extend in a completely analogous way to the case when the initial segments $(\mu_i)_0^n$ contain S zeros arranged arbitrarily, under the condition $\mu_0 = 0$.

Finally, let us note an application of the proved theorems to the problem of V. A. Markov ([2]) on finding a polynomial $Y_n(x)$ least deviating from 0 on $[0,1]$ among those whose coefficients (y_i) are connected by the relation

$$\sum_{i=0}^{n} \mu_i y_i = A \ (\neq 0). \tag{6}$$

Since this problem is identical to the problem of finding the extremal polynomial $Q_n(x)$ of the segment-functional $(\mu_i)_0^n$ ([1]), we immediately obtain:

1. Whatever the relation (6), containing l coefficients (but not containing y_0), $Y_n(x)$ has in its active distribution not fewer than $n+1-l$ alternations.
2. If, in addition, (6) contains $l < n/2$ coefficients, then the solution of Markov' s problem is unique.

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CITED LITERATURE

1. E. V. Voronovskaya, *The Method of Functionals and Its Applications*, Leningrad, 1963.

2. V. A. Markov, *On Functions Least Deviating from Zero*, 1892.

Note: Figure translations are in progress. See original paper for figures.