# BANACH COMMUTATIVE SYMMETRIC ALGEBRAS OF OPERATORS IN THE PONTRYAGIN SPACE \(\Pi_1\)

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## Abstract

## Full Text

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# BANACH COMMUTATIVE SYMMETRIC ALGEBRAS OF OPERATORS IN THE PONTRYAGIN SPACE $\Pi_1$

*(Presented by Academician I. M. Vinogradov, 15 VI 1967)*

Let $R$ be a commutative symmetric algebra of bounded operators in the Pontryagin space $\Pi_1$. In the works of M. A. Naimark ([1],[2] ) a classification of such algebras was given. It was shown that, under the assumptions of separability of $R$ and $\Pi_1$, all such algebras (with the exception of certain simple cases) are determined, up to equivalence, by means of a certain model. There the conditions for equivalence of algebras defined by such a model were also found. In fact, two distinct types of inequivalent models were indicated: a) and b) (see ([1])); the algebras corresponding to them will be called, respectively, **singular** and **nonsingular**.

The operators of a singular algebra $R$ are given by the formulas

$$A\xi_0 = \lambda(A)\xi_0,$$

$$Ap(t) = (p, p_{A^*})\xi_0 + A_1 p = \tag{1}$$

$$= -\int_{T_1} (A_1(t) - \lambda(A))(p(t), \xi(t))\, d\sigma \cdot \xi_0 + A_1(t)p(t),$$

$$Aq = (q, q_{A^*})\xi_0 + \lambda(A)q,$$

$$A\eta_0 = \gamma(A)\xi_0 + \lambda(A)\eta_0 + p_A + q_A =$$

$$= \gamma(A)\xi_0 + \lambda(A)\eta_0 + (A_2(t) - \lambda(A))\xi(t) + q_A,$$

where $T$ is a bicompact space, $\sigma$ is a regular Borel measure on $T$, and $\lambda(A) = A_1(t_0)$ and $T_1 = T - \{t_0\}$, $t_0 \in T$. The point $t_0 \in T$ will be called a **singular point** of the singular algebra $R$.

For the operators of a nonsingular algebra $R$ we have formulas of the form

$$A\xi_0 = \lambda(A)\xi_0,$$

$$Ap(t) = (p, p_{A^*})\xi_0 + A_1 p =$$

$$= -\int_T (A_1(t) - \lambda(A))(p(t), \xi(t))\, d\sigma \cdot \xi_0 + A_1(t)p(t), \tag{2}$$

$$A\eta_0 = \gamma(A)\xi_0 + \lambda(A)\eta_0 + p_A =$$

$$= \gamma(A)\xi_0 + \lambda(A)\eta_0 + (A_1(t) - \lambda(A))\xi(t),$$

with now $\lambda(A) \neq A_1(t_0)$ for any point $t_0 \in T$. (For a detailed explanation of these formulas and of the notation connected with them, see ([2]).)

Thus, each operator $A \in R$ is defined by the system of parameters $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$. When $A$ runs through the algebra $R$, $A_1(t)$ runs through a certain symmetric algebra of functions $R_1 \subset C(T)$, uniformly dense in $C(T)$; the vectors $q_A$ form a certain linear manifold $\mathcal{L}$, and $\gamma(A)$ and $\lambda(A)$ are linear functionals on $R$ (see ([2])). The totality $\mathcal{B}$ of all systems $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$ corresponding to a certain algebra $R$ forms a linear manifold in the totality $G$ of all systems $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$, when all parameters independently run through their ranges of values. This manifold is called the **defining manifold** of the algebra $R$. The problem

of the present work is to describe the determining manifolds of complete singular and nonsingular algebras with identity.

1. Denote

$$M_0 = \{A :\ A \in R,\ \lambda(A) = 0\}.$$

Then, since $\lambda(A) = \overline{\lambda(A^*)}$, $M_0$ is a symmetric maximal ideal of the algebra $R$. Moreover, $R = 1 + M_0$, since for $B = A - \lambda(A) \cdot 1$ we have $\lambda(B) = 0$ and $B \in M_0$.

Denote $M_1 = \{A :\ A \in R,\ A_1(t) \equiv 0\}$. Then, since the homomorphism $A \to A_1(t)$ is continuous and symmetric (see (2)), $M_1$ is a closed symmetric ideal in $R$.

Denote also by $\widetilde{M}_0$ the image of the ideal $M_0$ under the homomorphism $A \to A_1(t)$. $\widetilde{M}_0$ is a symmetric maximal ideal in $R_1$, proper or improper. The following assertions are valid with respect to $\widetilde{M}_0$.

I. *If $R$ is nonsingular, then the ideal $\widetilde{M}_0$ is uniformly dense in $C(T)$.*

II. *If $R$ is singular, then the ideal $\widetilde{M}_0$ is uniformly dense in $C_{t_0}(T)$.*

2. Every operator of the algebra given by formulas (1) or (2) is bounded. The operator norm in the algebra can be expressed in terms of the parameters of $R$ via the kernel of the homomorphism $R \to R_1$ isomorphic to the algebra $R_1$, i.e. the operator norm up to equivalence has the form

$$\|A\| = |A_1| + |\gamma(A)| + |p_A| + |q_A| = \sup_{t\in T} |A_1(t)| + |\gamma(A)| + |q_A| +$$

$$+ \left[\int_{T_1} |A_1(t) - A_1(t_0)|^2 \, d\mu\right]^{1/2}, \qquad (3)$$

where $d\mu = (\xi(t), \xi(t))\, d\sigma$.

For a nonsingular algebra we have

$$\|A\| = |\lambda(A)| + |A_1| + |\gamma(A)| =$$

$$= |\lambda(A)| + |\gamma(A)| + \sup_{t\in T} |A_1(t)|. \qquad (4)$$

3. All the preceding assertions are valid in the case of an arbitrary algebra, complete or incomplete. We now impose the condition of completeness on the algebra $R$. Since the algebra $R_1$ is homomorphic to the algebra $R$, the factor algebra of $R$ by the kernel of the homomorphism $R \to R_1$ is isomorphic to the algebra $R_1$, i.e. the algebra $R/M_1$ is algebraically isomorphic to the algebra $R_1$. Since the homomorphism $R \to R_1$ is continuous and, consequently, the ideal $M_1$ is closed, by a known theorem (see, for example, (3), Ch. I, § 4.3) and the completeness of $R$ in the norm $\|A\|$, the algebra $R/M_1$ and, consequently, the isomorphic algebra $R_1$, will be complete in the natural norm in the quotient space $R/M_1$. If $\widetilde{A}$ is the class of elements from $R$ congruent modulo $M_1$ and $A_1(t) \in R_1$ is the function corresponding to it, then this natural norm is equal to

$$\|\widetilde{A}\| = \|A_1\| = \|A_1(t)\| = \inf_{A\in\widetilde{A}} \|A\|.$$

From formulas (3) and (4) it follows that this norm has the form

$$\|\widetilde{A}\| = \sup_{t \in T} |A_1(t)| + \inf_{A \in \widetilde{A}} (|\lambda(A)| + |\gamma(A)|) \tag{5}$$

for a nonsingular algebra $R$, and

$$\|\widetilde{A}\| = \sup_{t \in T} |A_1(t)| +$$

$$+ \inf_{A \in \widetilde{A}} \left[ |\lambda(A)| + |\gamma(A)| + |q_A| + \left( \int_{T_1} |A_1(t) - A_1(t_0)|^2 \, d\mu \right)^{1/2} \right] \tag{6}$$

for a singular algebra $R$.

Thus, in the case of completeness of $R$ in its norm, the algebra $R_1$ is not only uniformly dense in $C(T)$, but also complete in the norm (5) or (6). This fact makes it possible to describe concretely the structure of the algebra $R_1$ and of the determining manifolds for complete algebras.

4. Applying the arguments of Sec. 3 and taking assertion I into account, we obtain that, in the case of a complete nonsingular algebra, $R_1 = C(T)$, and for the defining manifold the following holds.

**Theorem 1.** *Every nonsingular complete algebra $R$ is equivalent to an algebra given by the formulas:*

$$A\xi_0 = \lambda\xi_0,$$

$$Ap(t) = A_1(t)p(t),$$

$$A\eta_0 = \gamma\xi_0 + \lambda\eta_0,$$

where $\lambda$ and $\gamma$ are arbitrary complex numbers, and $A_1(t)$ is an arbitrary function from $C(T)$.

5. In the study of singular algebras it is useful to consider the following collection of functions from $R_1$. Denote by $S$ the linear span of all possible products of functions from $R_1$. $S$ is a symmetric ideal in $R_1$. For $S$ the following holds.

**Lemma.** *The ideal $S$ is dense, in the norm*

$$\sup_{t \in T} |A_1(t)| + \left( \int_{T_1} |A_1(t)|^2 \, d\mu \right)^{1/2}$$

*in the ring of all continuous functions on $T$ that vanish at 0 at the point $t_0$, for which this norm is finite.*

A detailed consideration of the defining manifolds of complete singular algebras, using assertion II, the arguments of Sec. 3, and the lemma, leads to the fact that in this case $R_1$ consists of all continuous functions on $T$ for which the norm is finite:

$$\|A_1\| = \sup_{t \in T} |A_1(t)| + \left( \int_{T_1} |A_1(t) - A_1(t_0)|^2 \, d\mu \right)^{1/2}. \tag{7}$$

For defining manifolds of complete singular algebras the following holds.

**Theorem 2.** *Every complete singular algebra is given by a defining manifold of the form*

$$\{A_1(t), q_A, \gamma, \lambda(A) = A_1(t_0)\},$$

*where $\gamma$, $q_A$, and $A_1(t)$ independently run through, respectively: the set of complex numbers, a certain complete space $\mathcal{E}$, and the ring of all functions for which the norm (7) is finite.*

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*Note: Figure translations are in progress. See original paper for figures.*