EXTENSIONS OF MODULES

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Abstract

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B. M. RUDYK

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This paper is devoted to the study of short exact sequences in the category $\mathfrak{M}$. The objects of this category are modules (where the ring is not considered fixed), and the set of morphisms of a module (A, U) into a module (B, V) consists of pairs of mappings $\Phi = (\Phi_A, \Phi_U)$, where Φ_A (respectively Φ_U) is a homomorphism of the group A into the group B (of the ring U into the ring V), and

$$(a \cdot u)\Phi = a\Phi \cdot u\Phi, \qquad a\Phi = a\Phi_A, \quad a \in A, \qquad u\Phi = u\Phi_U, \quad u \in U.$$

Such pairs of mappings will be called homomorphisms of the module (A, U) into the module (B, V).

The category $\mathfrak{M}$ contains, as full subcategories, the category of abelian groups, the category of associative rings, and the category of modules over a fixed ring Λ, in which the morphisms are semilinear transformations.

If $\Phi = (\Phi_A, \Phi_U)$ is a homomorphism of the module (A, U) into the module (B, V), then $(A\Phi_A, U\Phi_U)$ is called the image of the module (A, U). A pair of sets (A_1, U_1) is called the kernel of the homomorphism Φ if $A_1 = \ker \Phi_A$, $U_1 = \ker \Phi_U$.

An ideal of the module (A, U) is a pair (A_1, U_1), where A_1 (respectively U_1) is a subgroup of the group A (an ideal of the ring U), $A \cdot U_1 \subseteq A_1$, $A_1 \cdot U \subseteq A_1$. Ideals, and only they, serve as kernels of homomorphisms. If (A_1, U_1) is an ideal of the module (A, U), then $(A/A_1, U/U_1)$ is called the factor module; here

$$(a + A_1) \cdot (u + U_1) = a \cdot u + A_1, \qquad a \in A, \ u \in U.$$

By $(\mathfrak{A}, \mathfrak{U})$ we shall denote the module of multiplications of the module (A, U). In particular, if (A, U) is an abelian group (respectively a ring), then $(\mathfrak{A}, \mathfrak{U})$ is the ring of endomorphisms (the ring of bimultiplications ([1])). There is a natural homomorphism

$$i : (A, U) \to (\mathfrak{A}, \mathfrak{U}).$$

The kernel of this homomorphism will be called the annihilator of the module (A, U) and denoted by (A_0, U_0). The image $(A^{\otimes}, U^{\otimes})$ of the module (A, U) under the homomorphism i is contained in the module $(\mathfrak{A}, \mathfrak{U})$ as an ideal.

An extension (C, W) of the module (A, U) by means of the module (B, V) induces a homomorphism

$$\theta : (B, V) \to (\mathfrak{A}/A^{\otimes}, \mathfrak{U}/U^{\otimes}).$$

Every homomorphism of (B, V) into $(\mathfrak{A}/A^{\otimes}, \mathfrak{U}/U^{\otimes})$ possessing certain additional properties will be called a coupling.

Equivalent extensions correspond to one and the same coupling homomorphism, which makes it possible to formulate the extension problem in the usual way:

1) to describe, up to equivalence, all extensions of the module (A, U) by means of the module (B, V) with a given coupling homomorphism;
2) to solve the problem of when there exists an extension (C, W) of the module (A, U) by means of the module (B, V) that induces the given coupling homomorphism.

Studying extensions of modules in the category $\mathfrak{M}$ with a given coupling homomorphism is of interest, apparently, also because various extensions of algebraic objects that have already appeared in the literature are contained here as special cases. Let us list some of them: 1) abelian extensions of abelian groups; 2) exten-

of associative rings; 3) extensions of the ring of operators of a module. Other more or less interesting cases also fall under this scheme. Let us note, for example, extensions of modules in the category of modules over a fixed ring Λ, in which the morphisms are semilinear transformations.

A module (A, U) is called **abelian** if U is a ring with zero multiplication and $a \cdot u = 0$ for any elements $u \in U$ and $a \in A$. The annihilator of a module is an abelian module.

Let (A, U) and (B, V) be arbitrary modules, and let a accompanying homomorphism

$$\theta : (B, V) \to (\mathfrak{A}/A^{\otimes},\ \mathfrak{U}/U^{\otimes})$$

be given. The homomorphism θ determines, up to equivalence, an extension $(C^{\times}, W^{\times})$ of the module $(A^{\times}, U^{\times}) = (A/A_0, U/U_0)$ by means of the module (B, V), called a θ-module. On the other hand, θ induces accompanying homomorphisms

$$\Phi_1 : (A^{\times}, U^{\times}) \to (\mathfrak{A}_0, \mathfrak{U}_0)$$

and

$$\Phi_2 : (B, V) \to (\mathfrak{A}_0, \mathfrak{U}_0),$$

which are glued together into the accompanying homomorphism

$$\Phi : (C^{\times}, W^{\times}) \to (\mathfrak{A}_0, \mathfrak{U}_0),$$

where $(\mathfrak{A}_0, \mathfrak{U}_0)$ is the module of multiplications of the module (A_0, U_0) (let us note that under the natural homomorphism the module (A_0, U_0) is mapped onto the zero submodule of the module $(\mathfrak{A}_0, \mathfrak{U}_0)$). Denote by $\mathfrak{X}^*$ the set of equivalence classes of extensions of the module (A_0, U_0) by means of the module $(C^\times, W^\times)$ with accompanying homomorphism Φ. Since on the set of equivalence classes of extensions of an abelian module by means of an arbitrary module with a given accompanying homomorphism one can naturally define the structure of an abelian group, which we shall call the **group of extensions**, a structure of an abelian group is defined on $\mathfrak{X}^*$.

We shall say that the module (A, U) is a $(B, V)^\otimes$-module if an accompanying homomorphism

$$\theta : (B, V) \to (\mathfrak{A}/A^\otimes,\, \mathfrak{U}/U^\otimes)$$

is given. In a natural way a $(B, V)^\otimes$-homomorphism of $(B, V)^\otimes$-modules is defined, which endows the class of $(B, V)^\otimes$-modules with the structure of a category. A $(B, V)^\otimes$-homomorphism of $(B, V)^\otimes$-modules is a homomorphism of modules, and therefore a $(B, V)^\otimes$-extension of modules is an extension of them.

The structure of a $(B, V)^\otimes$-module on (A, U), which is given by the accompanying homomorphism θ, induces structures of $(B, V)^\otimes$-modules on the annihilator (A_0, U_0) and on the quotient module $(A^\times, U^\times)$, and moreover (A, U) is their $(B, V)^\otimes$-extension with zero accompanying homomorphism. Denote by $\mathfrak{X}_\theta$ the set of equivalence classes of $(B, V)^\otimes$-extensions of the $(B, V)^\otimes$-module (A_0, U_0) by means of the $(B, V)^\otimes$-module $(A^\times, U^\times)$ with zero accompanying homomorphism. By (A, U) denote that one of these classes which contains the module (A, U). On $\mathfrak{X}_\theta$ one can naturally define the structure of a partial universal algebra with zero*.

There exists a homomorphism π of the group $\mathfrak{X}^*$ into $\mathfrak{X}_\theta$. We shall denote the image of the group $\mathfrak{X}^*$ in $\mathfrak{X}_\theta$ by $\mathfrak{X}^*_\theta$.

In the paper ([1]) an example was constructed of an accompanying homomorphism θ of a ring V into the ring of bimultiplications $\mathfrak{U}$ of a ring U, which cannot be induced by any extension of the ring U by means of the ring V.

Theorem 1. *An accompanying homomorphism*

$$\theta : (B, V) \to (\mathfrak{A}/A^\otimes,\, \mathfrak{U}/U^\otimes)$$

is induced by some extension of the module (A, U) by means of the module (B, V) if and only if $(A, U) \in \mathfrak{X}^*_\theta$.

Denote by $\mathfrak{X}^*_0$ the kernel of the homomorphism

$$\pi : \mathfrak{X}^* \to \mathfrak{X}^*_\theta.$$

Theorem 2. *Between all inequivalent extensions of the module (A, U) by means of the module (B, V) that induce the accompanying homomorphism θ, and the elements of the group $\mathfrak{X}^*_0$, there is a one-to-one correspondence. The group $\mathfrak{X}^*_0$ is isomorphic to the group of extensions of the module (A_0, U_0) by means of the module (B, V) with accompanying homomorphism Φ_2.*

Corollary 1. *Inequivalent extensions of the module (A, U) with zero annihilator by means of the module (B, V) are in one-to-one*

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* *Note added in proof.* One may assume that $\mathfrak{X}_\theta$ is an abelian group.

in a one-to-one correspondence with the various accompanying homomorphisms of the module (B, V) into the module $(\mathfrak{A}/A^\otimes, \mathfrak{U}/U^\otimes)$.

In the case of associative rings the result of Corollary 1 was obtained by S. Mac Lane ([1]).

Let an accompanying homomorphism θ of the module (B, V) into the module $(\mathfrak{A}/A^\otimes, \mathfrak{U}/U^\otimes)$ be given. Consider the exact sequence

$$(0,0) \to (F,R) \to (E,S) \to (B,V) \to (0,0),$$

where (E, S) is a free module, the definition of which is introduced so that (E, S) in the category of abelian groups (respectively, rings) is a free abelian group (free ring).

Denote by

$$\Delta = \Delta^{(E,S)}_{(F,R),(A,U)}$$

the abelian group of (E, S)-homomorphisms of the module (F, R) into the abelian module (A, U). The crossed homomorphisms of the module (E, S) into the module (A, U) also form an abelian group, which is homomorphically mapped into the group Δ. Its image in Δ will be denoted by

$$\Gamma^{(E,S)}_{(F,R),(A,U)}.$$

Theorem 3. *The group of extensions of the abelian module (A, U) by means of the module (B, V) with accompanying homomorphism θ is isomorphic to the factor group*

$$\Delta^{(E,S)}_{(F,R),(A,U)} \Big/ \Gamma^{(E,S)}_{(F,R),(A,U)}.$$

In the case of abelian groups Theorem 3 was obtained in the work of Eilenberg and Mac Lane ([2]).

An accompanying homomorphism $\theta : (B, V) \to (\mathfrak{A}/A^\otimes, \mathfrak{U}/U^\otimes)$ is called **semidirect** if θ is a module $(C^\times, W^\times)$ decomposable into the semidirect sum of the modules $(A^\times, U^\times)$ and (B, V), i.e. $(A^\times, U^\times)$ is an ideal of the module $(C^\times, W^\times)$, (B, V) is a submodule of the module $(C^\times, W^\times)$,

$$\{(A,U),(B,V)\} = (C^\times, W^\times)$$

and

$$(A,U) \cap (B,V) = (0,0).$$

Theorem 4. *On the set of classes of equivalent extensions of the module (A, U) by means of (B, V) with semidirect accompanying homomorphism θ, one can define in a natural way the structure of an abelian group, which is isomorphic to the group of extensions of the annihilator (A_0, U_0) of the module (A, U) by means of (B, V) with accompanying homomorphism Φ_2.*

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Moscow State University
named after M. V. Lomonosov

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References

[1] S. Mac Lane, Ill. J. Math., **2**, 316 (1958).
[2] S. Eilenberg, S. Mac Lane. Ann. Math., **43**, 757 (1942).

Note: Figure translations are in progress. See original paper for figures.