ON DEGENERATING LINEAR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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ON DEGENERATING LINEAR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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In a Banach space E consider the equation

$$A(t)\,du/dt + B(t)u = f(t), \tag{1}$$

where $0 < t < d$, $B(t)$ is a linear operator in E, bounded for all $t \in [0, d]$. With respect to the linear operator $A(t)$ we assume that for each $t \in (0, d]$ there exists a bounded inverse $A^{-1}(t)$, and that the function $a(t) = \|A^{-1}(t)\|^{-1}$ ($\| \;\|$ is the norm in E) belongs to $C^1[0, d]$ and $a(+0) = 0$.

If the operator $a(t)A^{-1}(t)$ is applied to both sides of equation (1), then equation (1) can be reduced to an equation of the form

$$a(t)\,du/dt + B(t)u = f(t), \tag{2}$$

where $B(t)$ is again a linear bounded operator in E, $a(t) \in C^1[0, d]$, $a(t) > 0$ for $t > 0$, and $a(+0) = 0$. We shall be interested in the question of the existence of solutions of equations (1) and (2) in E that are smooth up to the point $t = +0$. We use mainly the Hölder spaces of abstract functions $u(t)$ with values in E, $C^{m+\alpha}(E)$ ($m \geq 0$ an integer, $0 \leq \alpha < 1$), and also the C. L. Sobolev spaces $W_p^m(E)$ ($m \geq 0$ an integer, $1 \leq p < \infty$). The spaces corresponding to the spaces $C^{m+\alpha}(E)$, on the set of operator-functions $B(t)$ continuous on $[0, d]$ and acting in E, will be denoted by $C^{m+\alpha}(\mathcal{E})$.

1. For the homogeneous equation (2) it is easy to construct the resolving operator $U(t, s)$ ($0 \leq t, s \leq d$) in E. Suppose that the operator $B(0)$ in equation (2) satisfies one of the following conditions:

I. The spectrum of the operator $B(0)$ lies in the half-plane $\operatorname{Re}\lambda < -\beta$.

II. The spectrum of the operator $B(0)$ lies in the half-plane $\operatorname{Re}\lambda > \beta$.

Any solution of equation (2) can be represented in the form $u(t) = U(t, d)x + Kf(t)$, where $x \in E$ and the operator K is defined by the formula

$$Kf(t) = -\int_t^d U(t,s)f(s)\,\frac{ds}{a(s)}. \tag{3}$$

If condition II is satisfied with $\beta > 0$, one of the solutions of equation (2) has the form

$$u(t) = Lf(t) = \int_0^t U(t,s)f(s)\,\frac{ds}{a(s)}. \tag{4}$$

Theorem 1. *Let condition I be satisfied with a constant $\beta > \alpha a'(0)$, where $0 \le \alpha < 1$, $a(t) \in C^1[0,d]$. If $B(t) \in C^\alpha(\mathcal{E})$, then the operators $U(\cdot,d)$ and K act and are bounded in $C^\alpha(E)$.*

Theorem 2. *Let condition II be satisfied with a constant $\beta > 0$. If $0 \le \alpha < 1$, $a(t) \in C^1[0,d]$, and $B(t) \in C^\alpha(\mathcal{E})$, then the operator L acts and is bounded in $C^\alpha(E)$.*

The proofs of Theorems 1 and 2 are analogous to the proofs of the corresponding theorems in ([1]) and are based on the results of ([2]). On the basis of Theorems 1 and 2, the following assertions can be proved.

Theorem 3. Suppose condition I is satisfied with a constant $\beta > (m+\alpha)a'(0)$, where $m \ge 0$ is an integer, $0 \le \alpha < 1$, and $a(t) \in C^{m+\alpha}[0,d] \cap C^1[0,d]$. If $B(t) \in C^{m+\alpha}(\mathcal{E})$ and $f(t) \in C^{m+\alpha}(E)$, then any solution of equation (2) on $(0,d)$ belongs to $C^{m+\alpha}(E)$.

Theorem 4. If condition II is satisfied with a constant $\beta > 0$, $a(t) \in C^{m+\alpha}[0,d] \cap C^1[0,d]$, $B(t) \in C^{m+\alpha}(\mathcal{E})$, $f(t) \in C^{m+\alpha}(E)$, $0 \le \alpha < 1$, $m \ge 0$ is an integer, then on $(0,d)$ there exists one and only one solution of equation (2) belonging to $C^{m+\alpha}(E)$. This solution is representable in the form (4).

Remark. Theorems 3 and 4 are also valid for equation (1), if in their statements one sets $a(t) = \|A^{-1}(t)\|^{-1}$ and replaces $B(t)$ by $a(t)A^{-1}(t)B(t)$.

Theorems 1-4 carry over to the case of the spaces $W_p^m(E)$. We give here analogues of Theorems 3 and 4.

Theorem 5. Suppose condition I is satisfied with a constant $\beta > (m-1/p)a'(0)$, where $p \ge 1$ and $m \ge 0$ is an integer. If $a(t) \in C^m[0,d] \cap C^1[0,d]$, $B(t) \in C^m(\mathcal{E})$, $f(t) \in W_p^m(E)$, then any solution of equation (2) on $(0,d)$ belongs to $W_p^m(E)$.

Theorem 6. Suppose condition II is satisfied with a constant $\beta > -\frac{1}{p}a'(0)$, where $p \ge 1$, $a(t) \in C^m[0,d] \cap C^1[0,d]$, $m \ge 0$ is an integer. If $B(t) \in C^m(\mathcal{E})$ and $f(t) \in W_p^m(E)$, then there exists one and only one solution of equation (2) on $(0,d)$ belonging to $W_p^m(E)$. This solution is representable in the form (4).

2. The function $a(t)$, by means of which equation (1) was reduced to equation (2), describes only the character of growth of the norm of the inverse

operator $A^{-1}(t)$. In many cases it is necessary to take into account the behavior of the operator $A^{-1}(t)$ in various subspaces of the space E. Below we give several variants of such considerations for equation (1).

Assume that in E there exist bounded projection operators P_1 and P_2 ($P_1 + P_2 = I$), and also functions $a_1(t) \in C^1[0,d]$ and $a_2(t) \in C^1[0,d]$, positive for $t > 0$, such that the operators $a_i(t)P_{iA}^{-1}(t)$ $(i = 1,2)$ are bounded operators in E for all $t \in [0,d]$. It follows from this that, for $t \in [0,d]$, the operators $B_i(t) = a_i(t)P_{iA}^{-1}(t)B(t)$ $(i = 1,2)$ are bounded in E. The operators $B_1(t)$ and $B_2(t)$ act in, and are bounded in, the subspaces $E_1 = P_1E$ and $E_2 = P_2E$, respectively.

Consider the following cases:

III. $a_1(0) = a_2(0) = 0$. The spectrum of the operator $B_1(0)$, as an operator in E_1, lies in the half-plane $\operatorname{Re}\lambda < -\beta_1$; the spectrum of the operator $B_2(0)$, as an operator in E_2, lies in the half-plane $\operatorname{Re}\lambda > \beta_2 > 0$.

IV. $a_1(0) = 0,\ a_2(0) > 0$. The spectrum of the operator $B_1(0)$, as an operator in E_1, lies in the half-plane $\operatorname{Re}\lambda < -\beta_1$.

V. $a_1(0) = 0,\ a_2(0) > 0$. The spectrum of the operator $B_1(0)$, as an operator in E_1, lies in the half-plane $\operatorname{Re}\lambda > \beta_1$.

Theorem 7. Suppose condition III is satisfied with a constant $\beta_1 > \alpha a_1'(0)$, $0 \le \alpha < 1$, $a_i(t) \in C^1[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^\alpha(\mathcal{E})$, $B_i(t) \in C^\alpha(\mathcal{E})$ $(i = 1,2)$, $f(t) \in C^\alpha(E)$. If the operators K_1 and L_2, constructed by means of formulas (3) and (4) from $B_1(t), a_1(t)$ and $B_2(t), a_2(t)$, respectively, satisfy the condition

$$\|K_1B_1\|_{C^\alpha} \times \|L_2B_2\|_{C^\alpha} < 1,$$

then there exists a family of solutions of equation (1) on $(0,d)$ belonging to $C^\alpha(E)$. The solutions of this family depend on an arbitrary element $x_1 \in E_1$. Any solution $u(t) \in C^0(E)$ of equation (1) on $(0,d)$ belongs to this family.

Corollary. Suppose condition III is satisfied with a constant $\beta_1 > (m+\alpha)a_1'(0)$, where $m \ge 0$ is an integer, $0 \le \alpha < 1$, $a_i(t) \in C^{m+\alpha}[0,d] \cap C^1[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^{m+\alpha}(\mathcal{E})$, $B_i(t) \in C^{m+\alpha}(\mathcal{E})$ $(i = 1,2)$, $f(t) \in C^{m+\alpha}(E)$. If the conditions of Theorem 7 are satisfied after replacing the operators B_1 and B_2 by the operators $B_{1,j}(t) = B_1(t) + ja_1'(t)$ and $B_{2,j}(t) = B_2(t) + ja_2'(t)$ $(0 \le j \le m)$, then there exists a family of solutions of equation (1) on $(0,d)$ belonging to $C^{m+\alpha}(E)$ and depending on

of an arbitrary element $x_1 \in E_1$. Every solution $u(t) \in C^0(E)$ of equation (1) on $(0,d)$ belongs to this family.

We shall say that an abstract function $u(t)$ with values in E belongs to $C_k^{m+\alpha}(E)$ $(-\infty < k < \infty)$ if $t^k u^{(j)}(t) \in C^\alpha(E)$ for all integers $j : 0 \le j \le m$. If the projections $u_1(t) \equiv P_1u(t)$ and $u_2(t) \equiv P_2u(t)$ $(P_1 + P_2 = I)$ of the function $u(t) \in E$ onto E_1 and E_2 belong respectively to $C_{k_1}^{m_1+\alpha_1}(E_1)$ and $C_{k_2}^{m_2+\alpha_2}(E_2)$, then we shall write

$$u(t) \in C_{k_1}^{m_1+\alpha_1}(E_1) \times C_{k_2}^{m_2+\alpha_2}(E_2).$$

Theorem 8. Suppose condition IV is satisfied with a constant $\beta_1 > (m+\alpha)a_1'(0)$, where $m \geq 0$ is an integer and $0 \leq \alpha < 1$. If $a_i(t) \in C^{m+\alpha}[0,d] \cap C^1[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^{m+\alpha}(\mathcal{E})$, $B_i(t) \in C^{m+\alpha}(\mathcal{E})$ $(i=1,2)$, and $f(t) \in C^{m+\alpha}(E)$, then every solution of equation (1) on $(0,d)$ belongs to $C^{m+\alpha}(E_1) \times C^{m+1+\alpha}(E_2)$.

Theorem 9. Suppose condition V is satisfied with a constant $\beta_1 > 0$; $m \geq 0$ is an integer, $0 \leq \alpha < 1$, $a_i(t) \in C^{m+\alpha}[0,d] \cap C^1[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^{m+\alpha}(\mathcal{E})$, $B_i(t) \in C^{m+\alpha}(\mathcal{E})$ $(i=1,2)$. If $f(t) \in C^{m+\alpha}(E)$, then there exists a family of solutions $u(t) \in C^{m+\alpha}(E_1) \times C^{m+1+\alpha}(E_2)$ of equation (1) on $(0,d)$, depending on an arbitrary element $x_2 \in E_2$. Every solution $u(t) \in C^0(E)$ of equation (1) on $(0,d)$ belongs to this family.

Theorem 10. Suppose there exists $k \geq \alpha$ $(0 \leq \alpha < 1)$ such that condition IV is satisfied with a constant $\beta_1 > (m+\alpha-k)a_1'(0)$, where $m \geq 0$ is an integer. If $a_1(t) \in C^{m+\alpha}[0,d] \cap C^{1+\alpha}[0,d]$, $a_2(t) \in C^{m+\alpha}[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^{m+\alpha}(\mathcal{E})$, $B_i(t) \in C^{m+\alpha}(\mathcal{E})$ $(i=1,2)$, $f(t) \in C_k^{m+\alpha}(E)$, then every solution of equation (1) on $(0,d)$ belongs to

$$C_k^{m+\alpha}(E_1) \times C_k^{m+1+\alpha}(E_2).$$

Theorem 11. Suppose there exists $k \leq 0$ such that condition V is satisfied with a constant $\beta_1 > ka_1'(0)$. If $m \geq 0$ is an integer, $a_1(t) \in C^{m+\alpha}[0,d] \cap C^{1+\alpha}[0,d]$, $a_2(t) \in C^{m+\alpha}[0,d]$, $a_i(t)P_{iA}^{-1}(t) \in C^{m+\alpha}(\mathcal{E})$, $B_i(t) \in C^{m+\alpha}(\mathcal{E})$ $(i=1,2)$, $f(t) \in C_k^{m+\alpha}(E)$, then there exists a family of solutions $u(t)$ of equation (1) on $(0,d)$, belonging to

$$C_k^{m+\alpha}(E_1) \times C^{m+1+\alpha}(E_2),$$

and such that $a_1(t)\,du_1/dt \in C_k^{m+\alpha}(E_1)$. The solutions of this family depend on the set of elements $x_2 \in E_2$ satisfying the condition

$$B_1(t)U_2(t,0)x_2 \in C_k^{m+\alpha}(E_1),$$

where $U_2(t,s)$ is the resolving operator for the equation

$$a_2(t)\,du_2/dt + B_2(t)u_2 = 0$$

in E_2. Every solution $u(t)$ of equation (1) on $(0,d)$ for which the conditions

$$u(t) \in C_k^{m+\alpha}(E_1) \times C^{m+1+\alpha}(E_2)$$

and $a_1(t)\,du_1/dt \in C_k^{m+\alpha}(E_1)$ are satisfied belongs to this family.

Analogous theorems can also be proved for the spaces $W_p^m(E)$. We also note that analogues of Theorems 10 and 11 in the case when conditions I, II, and III are satisfied are easy to obtain if, in equation (2), one makes the substitution of the unknown function $u(t) = t^{-k}w(t)$ and uses Theorems 3, 4, 7.

3. As an example of an application of the results obtained, consider the problem of the existence on $(0, d)$ of smooth solutions of a linear degenerate differential equation of order n with complex-valued coefficients and right-hand side

$$\mathcal{L}_n u = b_0(t)u^{(n)} + b_1(t)u^{(n-1)} + \dots + b_n(t)u = f(t), \qquad (5)$$

in which the coefficient $b_0(t)$ and, possibly, some of the subsequent coefficients vanish at $t = +0$. Various special cases of the theorem stated below were obtained in the works ([1], [3–5]). Without loss of generality, we shall assume that $b_0(t)$ is real and positive for $t > 0$.

Let there exist an integer $q \in [1, n]$ such that $a_1(t) \equiv b_0^{1/q}(t) \in C^1[0, d]$, and let the limits

$$\theta_i = \lim_{t \to +0} a_1^{i-q}(t) b_i(t) \qquad (1 \le i \le q). \qquad (6)$$

exist.

Consider the algebraic equation of degree q in λ

$$(\dots((\Delta_1 - \lambda + \theta_1)(\Delta_2 - \lambda) + \theta_2) \dots + \theta_{q-1})(\Delta_q - \lambda) + \theta_q = 0, \qquad (7)$$

where $\Delta_i = (i - q)a_1'(0)$ $(1 \le i \le q)$. Denote by λ^* and λ_* the roots of equation (7) with the largest and smallest real parts, and put

$$\mu^* = \operatorname{Re} \lambda^* (a_1'(0))^{-1}$$

$(\mu^* = +\infty\ (-\infty)$ when $\operatorname{Re} \lambda^* \ge 0$ $(\operatorname{Re} \lambda^* < 0)$ and $a_1'(0) = 0)$,

$$\mu_* = \operatorname{Re} \lambda_* (a_1'(0))^{-1}$$

$(\mu_* = +\infty\ (-\infty)$ when $\operatorname{Re} \lambda_* > 0$ $(\operatorname{Re} \lambda_* \le 0)$ and $a_1'(0) = 0)$.

Theorem 12. Let $0 \le \alpha < 1$, $m \ge 0$, and $q \in [1, n]$ be integers, $a_1(t) \equiv b_0^{1/q}(t) \in C^{m+1+\alpha}[0, d]$, and let the functions

$$a_1^{1-q}(t)b_1(t), \quad a_1^{2-q}(t)b_2(t), \dots, \quad a_1^{-1}(t)b_{q-1}(t), \quad b_q(t), \quad b_{q+1}(t), \dots, b_n(t)$$

belong to $C^{m+\alpha}[0, d]$.

1) If there exists a number $k \ge 0$ such that $k > \mu^* + m + \alpha$, $k \ge \alpha$ when $k \ne 0$, and $f(t) \in C_k^{m+\alpha}[0, d]$, then every solution $u(t)$ of equation (5) belongs to $C^{m+n-q+\alpha}[0, d]$, and

$$a_1^{q-n+j}(t)u^{(j)}(t) \in C_k^{m+\alpha}[0, d]$$

for $n - q + 1 \le j \le n$.

2) If there exists a number $k \leq 0$ such that $k < \mu_*$ and $f(t) \in C_k^{m+\alpha}[0, d]$, then there exists a family of solutions $u(t)$ of equation (5) on $(0, d)$, belonging to $C^{m+n-q+\alpha}[0, d]$, and for which

$$a_1^{q-n+j}(t) u^{(j)}(t) \in C_k^{m+\alpha}[0, d]$$

$(n - q \leq j \leq n)$. If $q = n$, this family consists of only one solution, and equation (5) has no other solutions satisfying the conditions

$$a_1^j(t) u^{(j)}(t) \in C_k^0[0, d] \qquad [0 \leq j \leq n-1].$$

For $q < n$, the solutions of this family depend on as many arbitrary constants c_{n-j} $(q + 1 \leq j \leq n)$ as there are independent parameters c_{n-j} satisfying the condition

$$\sum_{j=q+1}^{n} \left(\sum_{i=0}^{n-j} b_{n-i}(t) \frac{t^{n-q-i-1}}{(n-q-i-1)!} \right) c_{n-j} \in C_k^{m+\alpha}[0, d]. \tag{8}$$

For $k = 0$, condition (8) is fulfilled for arbitrary c_{n-j}, and every solution $u(t)$ of equation (5) for which

$$a_1^{q-n+j}(t) u^{(j)}(t) \in C^0[0, d] \qquad (n - q + 1 \leq j \leq n-1), \quad u(t) \in C^{n-q}[0, d],$$

belongs to this family. For $k < 0$, every solution $u(t)$ of equation (5) satisfying the conditions

$$a_1^{q-n+j}(t) u^{(j)}(t) \in C_k^{m+\alpha}[0, d] \qquad (n - q \leq j \leq n), \quad u(t) \in C^{n-q-1}[0, d],$$

belongs to this family, and for it (8) must be satisfied with $c_{n-j} = u^{(n-j)}(0)$.

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Note: Figure translations are in progress. See original paper for figures.