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Abstract

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ON SOME MULTIPLICATIVE INEQUALITIES AND THEIR APPLICATION TO LINEAR SINGULAR INTEGRAL OPERATORS

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Let the function $u(x_1, \dots, x_n)$ be given in the n-dimensional cube $D: \{a \le x_1, x_2, \dots, x_n \le b\}$.

Definition. By the class H_δ we shall mean the collection of functions $u(x_1, \dots, x_n)$, defined in D, if for any M and $\widetilde{M}$ from D the inequality holds

$$|u(M) - u(\widetilde{M})| \le R_u \sum_{i=1}^{n} |x_i - \tilde{x}_i|^\delta,$$

$$0 < \delta \le 1, \quad M = M(x_1, \dots, x_n), \quad \widetilde{M} = \widetilde{M}(\tilde{x}_1, \dots, \tilde{x}_n),$$

where R_u is the Hölder constant.

Theorem 1. *For the class H_δ, for any $p > 0$ there exists $g > 0$ such that*

$$\|u\|_C \le q \|u\|^{(n+\delta p)/(n+2\delta p)} \|u\|_p^{\delta p/(n+2\delta p)}, \tag{1}$$

where

$$\|u\|_C = \max_{M \in D} |u(M)|, \quad \|u\|_\delta = \|u\|_C + \sup_{M, \widetilde{M} \in D} \left\{ \frac{|u(M) - u(\widetilde{M})|}{\sum_{i=1}^{u} |x_i - \tilde{x}_i|^\delta} \right\},$$

$$\|u\|_p^p = \int_D |u(M)|^p \, dv, \quad dv = dx_1 \dots dx_n,$$

$$q \le l = \max \left\{ n(b-a)^{\alpha_1}/2^\delta ;\ 2^{n/p}/(b-a)^{\alpha_3} \right\}^{1/(1+\alpha_3)},$$

$$\alpha_1 = \delta^2 p/(n + \delta p), \quad \alpha_2 = \alpha_1/\delta, \quad \alpha_3 = n/p.$$

Proof. Let $M_0(x_1^0, \dots, x_n^0)$ be a fixed point from D and $u(M) \in H_\delta$ $(u(M) \neq 0)$. Consider the set

$$E_u^{(i)}(M_0) = \begin{cases} [x_i^0,\ x_i^0 + c], & \text{if } a \le x_i^0 \le (a+b)/2, \\ [x_i^0 - c,\ x_i^0], & \text{if } (a+b)/2 < x_i^0 \le b, \end{cases}$$

where

$$c = l_1 \|u\|_p^{\widetilde{\beta}}, \quad l_1 = (b-a)^{\alpha_2}/2\|u\|_C^{p/(n+\delta p)}, \quad \tilde{\beta} = p/(n+\delta p), \quad i = 1, 2, \dots, n.$$

It is not hard to notice that $E_u^{(i)}(M_0) \subset [a, b]$.

Denote by $E_u(M_0)$ the collection of points $\{x_1, \dots, x_n\}$, where each of the components respectively ranges over the sets $E_u^{(1)}(M_0), \dots, E_u^{(n)}(M_0)$. Then it is clear that $E_u(M_0) \subset D$. Consequently,

$$\|u\|_{L_p} \ge \int_{E_u(M_0)} |u(M)|^p \, dv = |u(M')|^p l_1^n \|u\|_{L_p}^{n\widetilde{\beta}},$$

$$M' \in E_u(M_0), \quad M' = \{\xi_1, \xi_2, \dots, \xi_n\}, \quad \xi_i \in E^{(i)}(M_0), \quad i = 1, \dots, n.$$

Thus,

$$|u(M)| \le l_1^{-n/p} \|u\|_{L_p}^{\alpha_2}.$$

Since $u(M) \in H_\delta$ and $|\xi_i - x_i^0| \le c$, we have

$$|u(M_0)| \le R_u \sum_{i=1}^{n} |\xi_i - x_i^0| + |u(\widetilde{M})| \le (R_u n l_1^\delta + l_1^{-n/p}) \|u\|_{L_p}^{\alpha_2},$$

where

$$R_u = \sup_{M',\, M'' \in D} \left\{ \frac{|u(M') - u(M'')|}{\displaystyle\sum_{i=1}^{n} |x_i' - x_i''|^\delta} \right\}.$$

If instead of l_1 we substitute its expression, we obtain (1).

Corollary. Let $\rho(x) \ge 0$ $(a \le x \le b)$, and let, for $p > 1$, the integral converge:

$$\int_a^b \rho^{1/(1-p)}(x)\,dx < \infty.$$

Then for any $u(x) \in H_\delta$ $(0 < \delta \leq 1)$ the following holds:

$$\|u\|_C \leq q' \|u\|_\delta^{(1+\delta)/(1+2\delta)} \|u\|_{L_p(\rho)}^{\delta/(1+2\delta)}; \tag{1'}$$

q' is a constant independent of $u(x)$.

Let Γ be a closed or open smooth Lyapunov curve in the plane of the complex variable. For functions $u(\tau)$ given on the contour Γ and belonging to the Hölder class H_δ $(0 < \delta \leq 1)$, the following holds.

Theorem 2. For any $u(\tau) \in H_\delta$ and for any $p > 0$, the inequality

$$\|u\|_C \leq q_1 \|u\|_\delta^{(1+\delta p)/(1+2\delta p)} \|u\|_{L_p}^{\delta p/(1+2\delta p)} \tag{2}$$

holds, where

$$\|u\|_C = \max_{\tau \in \Gamma} |u(\tau)|; \qquad \|u\|_{L_p}^p = \int_\Gamma |u(\tau)|^p\,ds,$$

$$\|u\|_\delta = \|u\|_C + \sup_{\tau_1, \tau_2 \in \Gamma} \left\{ \frac{|u(\tau_1) - u(\tau_2)|}{|\tau_1 - \tau_2|^\delta} \right\}, \qquad q_1 = \text{const}.$$

Inequalities of the type (1) and (2) have also been obtained in the case when $u(\tau)$ belongs to the class $H(\varphi)$ (for the definition see (1)).

For the Guseinov classes $H_{\alpha,\beta,\delta}$ (for the definition see (1)) the following result has been obtained:

Theorem 3. For any $u(x) \in H_{\alpha,\beta,\delta}$ $(a < x < b)$ and for any $p > 0$, there exists a constant number $q_2 > 0$ such that

$$\|u\|_{C(\rho)} \leq q_2 \|u\|_{\alpha,\beta,\delta,\delta_0}^{(1+\delta_0 p)/(1+2\delta_0 p)} \|u\|_{L_p(\rho_1)}^{\delta_0 p/(1+2\delta_0 p)}, \tag{3}$$

where

$$0 < \delta_0 \leq \delta, \qquad \rho_1(x) = (x-a)^{(\alpha+\delta)p}(b-x)^{(\beta+\delta)p},$$

$$\|u\|_{C(\rho)} = \sup_{a<x<b} \{|u(x)|\rho(x)\}, \qquad \rho(x) = (x-a)^{\alpha+\delta}(b-x)^{\beta+\delta},$$

$$\|u\|_{\alpha,\beta,\delta,\delta_0} = \|u\|_{C(\rho)} + \sup_{a<x,y<b} \left\{ \frac{|W(x) - W(y)|}{|x-y|^{\delta_0}} \right\}, \qquad W(x) = u(x)\rho(x).$$

Now consider linear singular operators with Hilbert and Cauchy kernels

$$S_1 u = -\frac{1}{2\pi} \int_{-\pi}^{\pi} K(x,s) u(s) \operatorname{ctg} \frac{s-x}{2} \, d\tau,$$

$$S_2 f = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t,\tau) f(\tau)}{\tau - t} \, d\tau,$$

where $K(x,s)$ is a 2π-periodic function in x and s, satisfying a Hölder condition with exponent $\delta_1 > 0$ in x and $\delta > 0$ $(\delta < \delta_1)$ in s, while $h(t,\tau)$ is a function defined on the closed Lyapunov contour Γ, satisfying the same condition as $K(x,s)$.

If we take into account that ([2,3]) the operators ${S_1}^*$ and S_2 are bounded in the spaces H_δ and L_p $(0 < \delta < 1,\ p > 1)$, then, by virtue of inequalities (1) and (2), we have

Theorem 4. *The operators S_1 and S_2 act from H_δ into H_δ and satisfy the inequalities*

$$\|S_1 u\|_C \le q_1^* \|u\|_\delta^{(1+\delta p)/(1+2\delta p)} \|u\|_{L_p}^{\delta p/(1+2\delta p)}, \tag{4}$$

$$\|S_2 f\|_C \le q_2^* \|f\|_\delta^{(1+\delta p)/(1+2\delta p)} \|f\|_{L_p}^{\delta p/(1+\delta p)}; \tag{5}$$

$$0 < \delta < 1; \qquad 1 < p; \qquad q_i^* = \text{const}; \qquad i = 1,2.$$

For the singular operator

$$S_{\widetilde{\alpha}} u = (x-a)^{\widetilde{\alpha}} (b-x)^{\widetilde{\alpha}} \int_a^b \frac{u(s)\, ds}{(s-a)^{\widetilde{\alpha}} (b-s)^{\widetilde{\alpha}} (s-x)}$$

it has been established ([4]) that it acts boundedly from $H_\delta^{0}{}^{**}$ into H_δ^0 $(0 < \delta < \tilde{\alpha})$. Therefore, by virtue of Khvedelidze' s theorem ([3]) and inequality (1), we prove

Theorem 5. *The operator $S_{\widetilde{\alpha}}$ acts from H_δ^0 into H_δ^0 and satisfies the inequality*

$$\|S_\alpha u\|_C \le q_3 \|u\|_\delta^{(1+\delta p)/(1+2\delta p)} \|u\|_{L_p}^{\delta p/(1+2\delta p)}, \tag{6}$$

where $0 < \delta < \tilde{\alpha},\ p > 1/(1-\tilde{\alpha}),\ q_3 = \text{const}.$

Finally, let us consider the operator

$$Su = \int_a^b \frac{u(s)}{s-x}\, ds,$$

which acts boundedly in the Banach space $H_{\alpha,\beta,\delta}$ with norm

$$\|u\|_{\alpha,\beta,\delta} = \sup_{a<x<b} \{|u(x)|\gamma(x)\} + \sup_{a<x,y<b} \left\{ \frac{|W(x)-W(y)|}{|x-y|^\delta} \right\},$$

$$W(x) = u(x)\gamma(x), \qquad \gamma(x) = (x-a)^{\alpha+\delta}(b-x)^{\beta+\delta}.$$

Theorem 6. *The operator S acts from $H_{\alpha,\beta,\delta}$ into $H_{\alpha,\beta,\delta}$ and satisfies the inequality*

$$\|Su\|_{C(\gamma)} \le q_4 \|u\|_{\alpha,\beta,\delta}^{(1+\delta p)/(1+2\delta p)} \|u\|_{L_p(\gamma)}^{\delta p/(1+2\delta p)}, \tag{7}$$

where

$$0 < \alpha+\delta, \qquad \beta+\delta < 1, \qquad p > \max\{1/(1-\alpha-\delta),\ 1/(1-\beta-\delta)\},$$

$$q_4 = \text{const}.$$

Now let us consider a sequence $\{u_n\} \in H_\delta$, satisfying the condition

$$\lim_{n,m\to\infty} \|u_n - u_m\|_\delta^{\alpha_1} \|u_n - u_m\|_{L_p}^{1-\alpha_1} = 0, \qquad p > 1,$$

$$\alpha_1 = (1+\delta p)/(1+2\delta p), \qquad 0 < \delta < 1. \tag{8}$$

By virtue of (1) ($n = 1$) or (2) or (3)*** (depending on where it is defined—

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* We mean 2π-periodic functions from H_δ.

** H_δ^0 contains all functions from H_δ that vanish at the endpoints of the interval $[a, b]$.

*** In this case $\{u_n(x)\} \in H_{\alpha,\beta,\delta}$, and under (8) one must understand

$$\lim_{n,m\to\infty} \|u_n - u_m\|^{\alpha_1}_{\alpha,\beta,\delta} \|u_n - u_m\|^{1-\alpha_1}_{L_p(\gamma)} = 0.$$

chosen $\{u_n\}$, this sequence is fundamental in the metric of the space of continuous functions (in the case (3) with weight $\rho(x) = (x-a)^{\alpha+\delta}(b-x)^{\beta+\delta}$). By virtue of the completeness of the space C there exists $u_0 \in C$ for which $\lim_{n\to\infty} \|u_n - u_0\| = 0$. The sets of all such u_0, corresponding in the sense of fundamentality to the right-hand sides of inequalities (2), (3), (4), and (6), will be denoted respectively by $C^p_{\delta,1}$, $C^p_{\delta,2}$, $C^p_{\delta,3}$, and $C^p_{\delta,4}$. For these sets, using inequalities (1), (2), (3), (4), (5), (6), (7), one proves

Theorem 7. *The sets $C^p_{\delta,1}$, $C^p_{\delta,2}$, $C^p_{\delta,3}$, and $C^p_{\delta,4}$ are invariant respectively with respect to the operators S_1, S_2, $S_{\tilde{\alpha}}$ ($\tilde{\alpha} < 1$), and S.*

We note that the Hölder or Huseynov classes corresponding to these sets are proper subsets of these sets.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.